Coupling of the continuum and semiclassical limit. Part I: convergence of eigenvalues

Matthias Keller[a], Lorenzo Pettinari[b], and Christiaan J. F. van de Ven[c]

[a]University of Potsdam, Department of Mathematics, Campus Golm, Haus 9 Karl-Liebknecht-Straße 24-25, 14476 Potsdam, Germany
[b]Dipartimento di Matematica, Università di Trento and INFN-TIFPA and INdAM, Via Sommarive 14, I-38123 Povo, Italy
[c]Friedrich-Alexander-Universität Erlangen-Nürnberg, Department of Mathematics, Cauerstraße 11, 91058 Erlangen, Germany

October 7, 2025

Abstract

We analyse the semiclassical d-dimensional Schrödinger operator in the continuum $H_S^{\mathrm{cont}}(\lambda_N) = -\frac{1}{2}\Delta + \lambda_N^2 V$ discretized on a mesh with spacing proportional to $1/N$. The semi-classical parameter λ_N is chosen as $\lambda_N = N^{1-\gamma}$, with $\gamma \in (-1,1)$, so that N governs both the semiclassical and continuum limit simultaneously. We prove that the corresponding discrete Schrödinger exhibits the same spectral asymptotics as $H_S^{\mathrm{cont}}(\lambda_N)$, in the regime $\lambda_N \gg 1$. Specifically, we prove that all eigenvalues of the discrete operator converge to those of the continuum $H_S^{cont}(\lambda_N)$, as $\lambda_N \to \infty$. Beyond this semi-classical domain, we further investigate the spectral asymptotics for $\gamma \in \mathbb{R} \setminus (-1,1)$, thereby establishing a comprehensive theory that fully characterizes the eigenvalue behavior across all possible values of $\gamma \in \mathbb{R}$.

1 Introduction

The first part of this work is devoted to the analysis of the energy asymptotics of a d-dimensional discrete Schrödinger operator in a *combined* semi-classical and continuum limit, *cf.* (1.7). Unlike other approaches in this context [5, 6, 7, 11], we investigate a wide range of scalings, interlacing the semiclassical parameter with the discretization one. This coupling yields a rich variety of asymptotic regimes for the eigenvalues. In particular, we are able to pinpoint the precise region in parameter space where the limiting behavior aligns with the semiclassical limit of a continuum Schrödinger operator in d dimensions. More precisely, we rigorously show that, in the semi-classical regime $\lambda_N \gg 1$, with $\lambda_N = N^{1-\gamma}$ for $\gamma \in (-1,1)$ the scaling parameter, all eigenvalues of the discrete Schrödinger operator exhibit the same asymptotic behavior as those of the continuum operator

$$H_S^{\mathrm{cont}}(\lambda_N) = -\frac{1}{2}\Delta + \lambda_N^2 V,$$

where the potential V is suitably chosen. This result confirms the validity and appropriateness of our scaling relation between the semi-classical parameter λ_N and the discretization mesh size $1/N$, showing that it effectively captures the classical limit of the continuous model through the discrete approximation. Further analysis of Agmon-type estimates for eigenfunctions are deferred to our forthcoming companion paper [9].

In the second part of the paper, we emphasize that our analysis on eigenvalue asymptotics extends beyond the semi-classical regime. It not only identifies the interval $(-1,1)$ as the true semiclassical domain, but also classifies four additional regions where the spectral behavior deviates significantly. Framed within this broader context, our study provides a unified understanding of the various limiting regimes and underscores the role of the parameter γ in determining the spectral asymptotics.

1.1 Semiclassical analysis and discrete Schrödinger operators

We recall the usual semi-classical analysis for a Schrödinger operator in the continuum:

$$H^{cont}(h) = -\frac{h^2}{2}\Delta + V \quad \text{on} \quad L^2(\mathbb{R}^d), \tag{1.1}$$

where Δ is the usual Laplacian on $\mathbb{R}$, and V is a potential defined as multiplication operator that satisfies appropriate assumptions. The parameter h is a scaled and dimensionless version of Planck's constant, describing a classical theory in the regime $h \to 0$. If we set $\lambda = 1/h$, we can rewrite (1.1) as

$$H^{cont}(h) = H(1/\lambda_N) = -\frac{1}{2\lambda^2}\Delta + V = \frac{1}{\lambda^2}(-\frac{1}{2}\Delta + \lambda^2 V). \tag{1.2}$$

We then define

$$H_S^{cont}(\lambda) := -\frac{1}{2}\Delta + \lambda^2 V. \tag{1.3}$$

Notice that the behavior of $H(h)$ ($h \to 0$) is closely related to the behavior of $H_S^{cont}(\lambda_N)$ ($\lambda_N \to \infty$). It is precisely the operator $H_S^{cont}(\lambda_N)$ studied by Simon in [15, 16] together with the time dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = H_S^{cont}(\lambda_N)\psi.$$

In this work we consider a discretized version of $H_S^{cont}(\lambda_N)$ defined in the following manner. We consider a uniform mesh size δ and introduce a discrete Schrödinger operator $H^\delta(\lambda)$ on $\ell^2(\delta\mathbb{Z})$, defined by

$$H^\delta(\lambda)f(x) = -\frac{1}{2\delta^2}\sum_{|x-y|=\delta}(f(y)-f(x)) + \lambda^2 V(x)f(x). \tag{1.4}$$

The continuum limit deals with the analysis of $H^{\delta}(\lambda)$ as $\delta \to 0$ at fixed λ. Details on this limit can be found in e.g. [5, 7]. In contrast, the classical limit concerns the regime $h \to 0$, at fixed δ (see e.g. [3]).

In this work, we focus on $\delta_N := 1/N$ and $\lambda_N := N^{1-\gamma}$ for $\gamma \in (-1,1)$ fixed. Thus, we do not take the limit $\delta \to 0$ independently of λ_N, but consider the single limit $N \to \infty$ obeying $\delta_N = 1/N$ and $\lambda_N = N^{1-\gamma}$. In other words, the limit we consider *contemporarily* describes the continuum as well as the semi-classical limit. With these choices at hand, we rewrite $H^{\delta}(\lambda_N)$ as $H(N)$, that is

$$H(N)f(x) = -\frac{N^2}{2} \sum_{|x-y|=\frac{1}{N}} (f(y) - f(x)) + N^{2(1-\gamma)} V(x) f(x), \quad \text{on} \;\; \ell^2(\mathbb{Z}^d/N). \tag{1.5}$$

The term $-N^2 \sum_{|x-y|=\frac{1}{N}} (f(y) - f(x)) = N^2 \sum_{|x-y|=\frac{1}{N}} (f(x) - f(y))$ is the discrete (necessarily positive) Laplacian Δ_{disc} (see e.g. [8]),

$$\Delta_{disc} f(x) := N^2 \sum_{|x-y|=\frac{1}{N}} (f(x) - f(y)) = N^2 \sum_{y \sim x} (f(x) - f(y)) \quad \text{on} \;\; \ell^2(\mathbb{Z}^d/N) \tag{1.6}$$

In other words, $H(N)$ now reads

$$H(N) = \frac{1}{2}\Delta_{disc} + N^{2(1-\gamma)} V \quad \text{on} \;\; \ell^2(\mathbb{Z}^d/N). \tag{1.7}$$

In Section 3 we provide an overview of the eigenvalue asymptotics of the other regimes of the parameter γ, starting from $H(N)$, *cf.* (1.7). In particular, we may identify a total number of five distinct limits with the following characteristics for the eigenvalues

- $\gamma > 1$: corresponding to the continuum limit of the free discrete Laplacian Δ_N, *cf.* (1.8), towards the Laplacian $-\frac{1}{2}\Delta$ on $L^2(\mathbb{R}^d)$;
- $\gamma = 1$: corresponding to the continuum limit of $H(N)$ towards the operator $H = -\frac{1}{2}\Delta + V$;
- $\gamma \in (-1,1)$: corresponding to a semi-classical limit of the operator $H_S^{cont}(\lambda)$, with relevant result stated in Theorem 1.2;
- $\gamma = -1$: corresponding to a purely discrete model, whose precise features depend on the particular choice of the potential;
- $\gamma < -1$: corresponding to a semiclassical approximation of a discrete model.

In particular, the regime $\gamma \in (-1,1)$ yields not only the semiclassical spectral asymptotics, but for this choice of γ it also allows to perform exact Agmon estimates for the ground state, as it has been done in [15] for continuum models. This constitutes the core focus of our second work.

1.2 Mathematical setting and notations

For practical purposes, for any $N \in \mathbb{N}_+$ we introduce the following Laplacian Δ_N on $\ell^2(\mathbb{Z}^d)$:

$$\Delta_N := N^2 \sum_{y \sim x} (f(x) - f(y)), \quad \text{on } \ell^2(\mathbb{Z}^d) \tag{1.8}$$

Notice that Δ_N on $\ell^2(\mathbb{Z}^d)$ precisely corresponds to Δ_{disc} on $\ell^2(\mathbb{Z}^d/N)$. With this notation at hand, the Hamiltonian $H(N)$, now seen as a sequence of operators on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$ becomes

$$H(N)f(x) := \frac{1}{2}(\Delta_N f)(x) + \lambda_N^2 V_N(x) f(x), \tag{1.9}$$

where $V_N : \mathbb{Z} \to \mathbb{Z}$ satisfies $V_N(x) := V(x/N)$ and V is a potential satisfying the following thre assumptions below. We will stick to this notation in the calculations presented in the subsequent sections, as it highlight the dependence on N, making the computation more explicit.

Assumption 1. *The following conditions on V are required:*

(1) $V \in C^\infty(\mathbb{R}^d)$ *and* $V \geq 0$.

(2) V *has* $1 \leq \kappa < +\infty$ *zeros ,* $a_i, \ldots a_\kappa$, *where*

$$M^{(a_i)}_{\alpha,\beta} := (\partial_{x_\alpha} \partial_{x_\beta} V)(a_i) \tag{1.10}$$

is strictly positive definite for all $i = 1 \cdots \kappa$*; i.e.* $(a_i)_{i=1}^\kappa$ *are non-degenerate minima for* V.

(3) V *is strictly positive at infinity, i.e. there exists a* $\delta > 0$ *and* $R_0 > 0$ *such that* $|x| > R_0$ *implies* $V(x) \geq \delta$.

NOTATION 1.1: The following notation is used throughout this paper.

- We will denote the generic Hamiltonian with potential satisfying Assumptions (1) by $H(N)$. The eigenfunctions and eigenenergies corresponding to $H(N)$ will be denoted by $\Omega_n(\cdot, N)$, respectively $E_n(N)$, where, as before, n labels the number of eigenstates/eigenenergies in increasing order.

- Thanks to the first two assumptions in 1, the Hessians $M^{(a_i)}$, can be diagonalized with strictly positive eigenvalues $\omega_\alpha(a_i)^2$, for all $1 \leq \alpha \leq d$ and $i = 1, \ldots, \kappa$. We define a set

$$\Big\{ \sum_{\alpha=1}^{d} \omega_\alpha(a_i)(m_\alpha + \frac{1}{2}),\, m_1, \ldots, m_d \in \mathbb{N},\, i = 1, \ldots, \kappa \Big\}, \tag{1.11}$$

  and we order its elements as $e_0(V) \leq e_1(V) \leq \ldots$.

- We introduce the d-dimensional discrete harmonic oscillator Hamiltonian

$$H^{harm}(N) = \frac{1}{2}\Delta_N + \lambda_N^2 V_N^{harm},$$

with

$$V_N^{harm}(x) := \sum_{\alpha=1}^{d} \frac{\omega_\alpha^2 x_\alpha^2}{2N^2},$$

for some arbitrary $\omega_\alpha > 0$, $\alpha = 1,\ldots,d$. The eigenfunctions and eigenenergies corresponding to the harmonic oscillator will be denoted by $\Omega_n^{harm}(\cdot,N)$, respectively $E_n^{harm}(N)$. Here $n = 0$ denotes the lowest eigenstate and lowest eigenenergy (i.e. the ground state), respectively; $n = 1$ the first excited eigenstate and eigenenergy; and so on. Coherently with Eq. (1.11) we will order the elements of

$$\{\sum_{\alpha=1}^{d} \omega_\alpha (m_\alpha + \frac{1}{2}),\ m_1,\ldots,m_d \in \mathbb{N}\},$$

as $e_0(V^{harm}) \le e_1(V^{harm}) \le \ldots$.

- For our analysis, *cf.* Section 2.2, we also need to introduce a modified oscillator, namely

$$\widetilde{H}(N) = \frac{1}{2}\Delta_N + \lambda_N^2 \sum_{\alpha=1}^{d} \widetilde{V}_{\alpha,N} \tag{1.12}$$

where

$$\widetilde{V}_{\alpha,N}(x) := \begin{cases} \dfrac{\omega_\alpha^2 x_\alpha^2}{2N^2} & \text{if } |x_\alpha| < \lfloor N^\Theta \rfloor \\ \dfrac{\omega_\alpha^4 x_\alpha^4}{2N^2} & \text{if } |x_\alpha| \ge \lfloor N^\Theta \rfloor \end{cases}, \tag{1.13}$$

for some $\Theta > 0$ to be specified at a later stage. The pertinent eigenfunctions and eigenenergies corresponding to $\widetilde{H}(N)$ will be denoted by $\widetilde{\Omega_n}(\cdot,N)$, respectively $\widetilde{E_n}(N)$.

- The floor function $\lfloor x \rfloor$ denotes the maximal integer lower than x, e.g. $\lfloor 2.3 \rfloor = 2$, while $\lfloor -0.7 \rfloor = -1$.

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1.3 Main result

As mentioned in the introduction the Hamiltonian we consider is of the form

$$H(N) = \frac{1}{2}\Delta_N + \lambda_N^2 V_N \quad \text{on } \ell^2(\mathbb{Z}^d), \tag{1.14}$$

with $\lambda_N = N^{1-\gamma}$, for some $\gamma \in \mathbb{R}$ and $V_N(x) := V(x/N)$, where V is required to satisfy the Assumptions 1. We will analyze the asymptotic behavior of the discrete eigenvalues of the latter operator as $N \to +\infty$, for all the possible values of γ in λ_N.

The main result of this paper consists in determining the correct interval of the parameter γ, to recover the correct semiclassical properties observed in continuum models (see [17]). Specifically, we will prove the following.

THEOREM 1.2: **(energy estimates)**
Let $H(N)$ be as in Eq. (1.14). Then, for $\gamma \in (-1,1)$, $H(N)$ has at least n-eigenvalues and it holds

$$\limsup_{N\to+\infty} \frac{E_n(N)}{\lambda_N} = e_n(V), \quad \text{for all } n \geq 0 \tag{1.15}$$

where $e_n(V)$ has been defined in Eq. (1.11).

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REMARK 1.3: Note that the energy levels $e_n(V)$ are precisely those obtained from a semi-classical limit starting with $H_S^{cont}(\lambda_N)$, with $\lambda_N = N^{1-\gamma}$ and $\gamma \in (-1,1)$, *cf.* [17, Thm. 1.1]. This indeed confirms that the coupled limit we take corresponds to the correct semi-classical regime of a Schrödinger operator in the continuum.

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To prove Theorem 1.2, in Section 2 we will first study a one dimensional harmonic oscillator and a one dimensional modified oscillator, obtaining upper (§ 2.1) and lower bounds (§ 2.2) respectively for the asymptotic values of their energies. These estimate, especially the lower bound one, will require the introduction of new mathematical techniques, such as the Agmon-Allegretto-Piepenbrink criterion 2.5 and clever ways to modify the Hamiltonians at play. In particular, the lower bound will be presented in the form of a theorem, i.e. Theorem 2.13. Subsequently, these bounds are extended to the d-dimensional case through a straightforward argument.

The oscillator Hamiltonian will come in hand when estimating a generic Hamiltonian like (1.14), since in the limit $N \to +\infty$, the eigenfunctions tend to localize around the minima of the potential, where the problem can be reduced to the one of studying an harmonic oscillator. This is discussed in § 2.3 and § 2.2.

In Section 3 we will discuss the eigenvalue asymptotics for $\gamma \in \mathbb{R}$, thereby extending our analysis for $\gamma \in (-1,1)$. In Sec. 4, we will discuss our results and possible extensions of this work. Appendix A presents the proof of proposition A.1.

2 Energy estimates

In this section we prove Theorem 1.2. To do this, we will start with a preliminary analysis for the one-dimensional harmonic oscillator. In particular, we will obtain the convergence of the ground state and first excited energy levels to the continuous harmonic oscillator ones.

2.1 Upper bound for the ground and first excited state of the harmonic oscillator

We focus here on proving an upper bound for the energy level of a one dimensional harmonic oscillator.

$$H^{harm}(N) = \frac{1}{2}\Delta_N + \lambda_N^2 V_N^{harm}(x), \quad \text{on } \ell^2(\mathbb{Z})$$

where

$$V_N^{harm}(x) = \frac{\omega^2}{2}\frac{x^2}{N^2}.$$

PROPOSITION 2.1:

Let $H^{harm}(N)$ as above, and $\lambda_N = N^{1-\gamma}$, $\gamma \in (-1,1)$. The following upper bound holds true

$$\limsup_{N\to\infty} \frac{E_0^{harm}(N)}{\lambda_N} \leq \frac{\omega}{2}.$$

$\diamond$

Proof. Let us introduce the normalized test function

$$\psi_0(x) := \frac{1}{C_N^{(0)}} e^{-\alpha\frac{x^2}{N^2}} J_N(x), \quad C_N^{(0)} := \left(\sum_{x\in\mathbb{Z}} e^{-2\alpha\frac{x^2}{N^2}} J_N(x)^2\right)^{1/2}, \tag{2.1}$$

where $\alpha \in O(\lambda_N)$ and J_N is a cut-off function, obeying

$$J_N(x) := \begin{cases} 1 & \text{if } |x| < \frac{N^{1+\delta}}{\lambda_N^{1/2}} \\ 0 & \text{otherwise} \end{cases}, \tag{2.2}$$

for some arbitrarily small $\delta > 0$.The coefficients $(C_N^{(0)})^2$ can be estimated as follows. We take N sufficiently large and estimate the sum by a Riemann integral from below, i.e.

$$\begin{aligned}(C_N^{(0)})^2 &= \sum_{x\in\mathbb{Z}} e^{-2\alpha\frac{x^2}{N^2}} J_N(x)^2 = 2\sum_{x\in\mathbb{Z}_+} e^{-2\alpha\frac{x^2}{N^2}} J_N(x)^2 - 1 \geq 2\int_0^\infty e^{-2\alpha\frac{x^2}{N^2}} J_N(x)^2 \mathrm{d}x - 1 \\ &= 2N\int_0^{N^\delta/\lambda_N^{1/2}} e^{-2\alpha x^2}\mathrm{d}x - 1 = \frac{2N}{(2\alpha)^{1/2}}\int_0^{N^\delta(2\alpha)^{1/2}/\lambda_N^{1/2}} e^{-y^2}\mathrm{d}y - 1,\end{aligned} \tag{2.3}$$

the latter following by a substitution of variables $y = (2\alpha)^{1/2}x$. Since $\alpha = O(\lambda_N)$

$$N^\delta(2\alpha)^{1/2}/\lambda_N^{1/2} \to \infty, \quad N \to \infty.$$

Hence, the right-hand side of the above integral can be bounded by

$$\begin{aligned}&\frac{2N}{(2\alpha)^{1/2}}\int_0^{N^\delta(2\alpha)^{1/2}/\lambda_N^{1/2}} e^{-y^2}\mathrm{d}y - 1 = \frac{2N}{(2\alpha)^{1/2}}\left(\int_0^\infty e^{-y^2}\mathrm{d}y - \int_{N^\delta(2\alpha)^{1/2}/\lambda_N^{1/2}}^\infty e^{-y^2}\mathrm{d}y\right) - 1 \geq \\ &\frac{N\pi^{1/2}}{(2\alpha)^{1/2}} - \frac{2N}{(2\alpha)^{1/2}}\frac{\lambda_N^{1/2}}{2N^\delta(2\alpha)^{1/2}} e^{-N^{2\delta}(2\alpha)/\lambda_N} - 1 = \frac{N\pi^{1/2}}{(2\alpha)^{1/2}} - O(e^{-DN^{2\delta}}) - 1,\end{aligned} \tag{2.4}$$

for some constant $D > 0$. In this computation, we have used that for any positive $c > 0$

$$\int_c^\infty e^{-y^2} = \int_c^\infty \frac{-1}{2y}(-2y)e^{-y^2} = \frac{1}{2c}e^{-c^2} - \int_c^\infty \frac{-1}{-2y^2}e^{-y^2}\mathrm{d}y \leq \frac{1}{2c}e^{-c^2},$$

i.e. we have taken $c = N^{\delta}(2\alpha)^{1/2}/\lambda_N^{1/2}$. Finally, we observe that

$$\frac{\lambda_N^{1/2}}{2N^{\delta}(2\alpha)^{1/2}} e^{-N^{2\delta}(2\alpha)/\lambda_N} = O(e^{-DN^{2\delta}}),$$

justifying (2.4). It therefore follows that

$$(C_N^{(0)})^2 \geq \frac{\pi^{1/2}N}{(2\alpha)^{1/2}} - 1 - O(e^{-DN^{2\delta}}), \quad \text{as } N \to \infty.$$

We now proceed to estimate $\langle \psi_0, x^2/N^2\psi_0\rangle$ from above. Take N sufficiently large so that in a similar fashion as done here above,

$$\begin{aligned}
(C_N^{(0)})^2\langle \psi_0, x^2/N^2\psi_0\rangle &= 2(C_N^{(0)})^2 \sum_{x\in\mathbb{Z}} \psi_0^2(x)x^2/N^2 \qquad (2.5)\\
&= 2(C_N^{(0)})^2 \sum_{x=1}^{\lfloor N/\lambda_N^{1/2}\rfloor - 1} \psi_0^2(x)x^2/N^2 + 2(C_N^{(0)})^2 \sum_{\lfloor N/\lambda_N^{1/2}\rfloor+1}^{\lfloor N^{\delta}/\lambda_N^{1/2}\rfloor} \psi_0^2(x)x^2/N^2 + O(\lambda_N^{-1})\\
&\leq 2N\int_0^{N^{\delta}/\lambda_N^{1/2}} e^{-2\alpha x^2}x^2\mathrm{d}x + O(\lambda_N^{-1}) \leq \frac{2N}{(2\alpha)^{3/2}}\int_0^{N^{\delta}(2\alpha)^{1/2}/\lambda_N^{1/2}} e^{-y^2}y^2\mathrm{d}x + O(\lambda_N^{-1})\\
&\leq \frac{N\sqrt{\pi}}{2(2\alpha)^{3/2}} + O(\lambda_N^{-1}). \qquad (2.6)
\end{aligned}$$

Note that the order $O(\lambda_N)$ is stricly smaller than $O(N/\alpha^{3/2})$ since for $\alpha \in O(\lambda_N)$ we have

$$\lambda_N \cdot \frac{\sqrt{\pi}}{2^{5/2}}\frac{N}{\alpha^{3/2}} \geq BN^{1/2+\gamma/2} \to +\infty, \quad \text{if } \gamma > -1. \qquad (2.7)$$

For the Laplacian, we estimate in a similar manner as above the following quantity

$$(C_N^{(0)})^2\langle \psi_0, \frac{1}{2}\Delta_N\psi_0\rangle = \frac{N^2}{2}\sum_{x\in\mathbb{Z}} e^{-\alpha\frac{x^2}{N^2}}J_N(x)\left(2e^{-\alpha\frac{x^2}{N^2}}J_N(x) - e^{-\alpha\frac{(x+1)^2}{N^2}}J_N(x+1) - e^{-\alpha\frac{(x-1)^2}{N^2}}J_N(x-1)\right) \qquad (2.8)$$

$$\begin{aligned}
&\leq \frac{N^2}{2}(1-e^{-\frac{\alpha}{N^2}}) + N^2N\int_0^{N^{\delta}/\lambda_N^{1/2}} e^{-2\alpha x^2}(2 - e^{-\alpha/N^2}e^{-2\alpha\frac{x}{N}} - e^{-\alpha/N^2}e^{2\alpha\frac{x}{N}})\mathrm{d}x\\
&\leq \frac{N^3\sqrt{\pi}}{(2\alpha)^{1/2}}(1-e^{-\frac{\alpha}{2N^2}}) + \frac{N^2}{2}(1-e^{-\frac{\alpha}{N^2}}) + O(e^{-DN^{2\delta}}). \qquad (2.9)
\end{aligned}$$

Again, since $-1 < \gamma < 1$ and $\alpha \in O(\lambda_N)$, we have $\alpha/N^2 \to 0$ as $N \to +\infty$, so that we can estimate the exponentials appearing in the last line of Eq. (2.8) by Taylor expansion. After combining all these estimates, we obtain

$$E_0^{harm}(N) \leq \langle \psi_0, \frac{1}{2}\Delta_N\psi_0 + \lambda_N^2 V^{harm}\psi_0\rangle \leq \frac{\alpha}{2} + \lambda_N^2\omega^2\frac{1}{8\alpha} + O(N^{1/2-3\gamma/2}). \qquad (2.10)$$

For large N this quantity is minimized by the value $\alpha = \lambda_N \omega/2$ which gives

$$E_0^{harm}(N) \leq \frac{1}{2}\omega\lambda_N + O(N^{1/2-3\gamma/2}), \tag{2.11}$$

resulting in

$$\limsup_{N\to\infty} \frac{E_0^{harm}(N)}{\lambda_N} \leq \frac{\omega}{2}. \tag{2.12}$$

$\square$

REMARK 2.2: **(Discrete limit)** We remark that the proof of Prop. 2.1 could be extended easily for $\gamma = 1$, i.e. $\lambda_N \in O(N^0)$, by removing the cut-off function J_N from Eq. (2.1). Setting $\gamma = 1$ corresponds to taking the purely continuum limit, without including the semiclassical one.

$\diamond$

REMARK 2.3: **(Excited energies)** Following a similar scheme as in 2.1, it is quite easy to obtain an estimate also for the nth excited energy level. We consider

$$\psi_n(x) \colon = \frac{1}{C_N^{(n}} J_N(x) H_n\big(x\frac{\sqrt{\lambda_N\omega}}{N}\big) e^{-\frac{\omega\lambda_N x^2}{2N^2}}, \quad C_N^{(n)} \colon = \Big(\sum_{x\in\mathbb{Z}} J_N^2(x) H_n\big(x\frac{\sqrt{\lambda_N\omega}}{N}\big)^2 e^{-\frac{\lambda_N\omega x^2}{N^2}}\Big)^{1/2}, \tag{2.13}$$

where $\mathbb{Z} \ni x \to H_n(x\sqrt{\lambda_N\omega}/N)$ is the discretization of the nth hermite polynomial. By simple estimates, using the orthogonality of Hermite polynomials one deduces that

$$\langle \psi_n, \psi_m \rangle = \delta_{n,m} + O(N^{1/2-3\gamma/2}), \tag{2.14}$$

$$\langle \psi_n, H^{harm}(N)\psi_m \rangle \leq \delta_{n,m}(n+\frac{1}{2})\omega\lambda_N + O(N^{1/2-3\gamma/2}). \tag{2.15}$$

To perform these latter estimates one has to divide $\mathbb{Z}$ in a finite number of regions depending on whether $\mathbb{Z} \ni x \to \psi_n(x)\psi_m(x)$, $\mathbb{Z} \ni x \to \psi_n(x)\Delta_N\psi_m(x)$ and $\mathbb{Z} \ni x \to \psi_n(x)V_N(x)\psi_m(x)$ are increasing or decreasing functions, to estimate the summation involved in Eqs. (2.14),(2.15) with the corresponding integrals. Then, using the Rayleigh-Riesz principle [19, Th. XIII.3], one obtains

$$\limsup_{N\to\infty} \frac{E_n^{harm}(N)}{\lambda_N} \leq \limsup_{N\to\infty} \frac{1}{\lambda_N}\langle \psi_n, H^{harm}(N)\psi_n \rangle \leq (n+\frac{1}{2})\omega. \tag{2.16}$$

$\diamond$

2.2 Lower bound for the ground and first excited state for a modified harmonic oscillator

In this section, we derive a lower bound for the ground state and first excited state energies of a slightly modified discrete harmonic oscillator. This analysis is motivated by the need to apply the Agmon–Allegretto-Piepenbrink criterion (see below), which provides a powerful framework for obtaining such estimates. However, applying this criterion directly to the standard harmonic oscillator is not straightforward. The main difficulty lies in the breakdown of the continuum approximation at large distances, which complicates the use of Agmon-type estimates.

To address this issue, we introduce a modified potential that preserves the properties of the original system: since the relevant quantum states are localized near the minima of the potential, any modification at sufficiently large distances does not alter the essential physics. Instead, it allows us to circumvent the limitations of the original potential at large scales.

We first recall the notion of a α-superharmonic function.

DEFINITION 2.4: For $\alpha \in \mathbb{R}$, we say that a function $u \in C(\mathbb{Z})$ is **α-subharmonic**, if

$$(L+\alpha)u \leq 0.$$

We say that u is **α-superharmonic** if $-u$ is α-subharmonic, i.e., u satisfies

$$(L+\alpha)u \geq 0.$$

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The Agmon–Allegretto-Piepenbrink criterion is now formulated as follows.

LEMMA 2.5: **(Agmon–Allegretto–Piepenbrink)**[8, Theorem 4.14]
Let $\alpha \in \mathbb{R}$ and consider a discrete Schrödinger operator on $\ell^2(\mathbb{Z}^d)$ with lowest eigenvalue $E_0(N)$. Then, the following statements are equivalent:

(i) $\alpha \geq -E_0(N)$.

(ii) There exists a strictly positive α-superharmonic function.

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Estimate for the ground state Once we have fixed a value of $\gamma \in (-1,1]$, we can prove:

PROPOSITION 2.6: Let $\delta > 0$ be such that $\gamma - 2\delta > -1$ and consider the one dimensional modified oscillator (1.1)

$$\widetilde{H}(N) = \frac{1}{2}\Delta_N + \lambda_N^2 \widetilde{V}_N \quad \text{on} \quad \ell^2(\mathbb{Z}),$$

where $\widetilde{V}_N$ is defined by

$$\widetilde{V}_N(x) := \begin{cases} \dfrac{\omega^2 x^2}{2N^2} & \text{if } |x| < \lfloor N^{1+\gamma-\delta} \rfloor \\ \dfrac{\omega^2 x^4}{2N^2} & \text{if } |x| \geq \lfloor N^{1+\gamma-\delta} \rfloor \end{cases}, \tag{2.17}$$

and $\omega > 0$ is a free parameter. Denote the ground state eigenvalue by $\widetilde{E}_0(N)$. Then,

$$\liminf_{N\to\infty} \frac{\widetilde{E}_0(N)}{\lambda_N} \geq \frac{\omega}{2}. \tag{2.18}$$

$\diamond$

Proof. To prove the result we will apply the Agmon-Allegretto-Piepenbrink criterion. To this end, we define $\Theta := 1+\gamma-\delta$ and consider the tentative supersolution

$$\psi(x) := \begin{cases} e^{-\lambda_N'\omega\frac{x^2}{2N^2}} & \text{if } |x| < \lfloor N^\Theta \rfloor \\ e^{-\lambda_N'\omega\frac{\lfloor N^\Theta\rfloor^2}{2N^2}} & \text{if } |x| \geq \lfloor N^\Theta \rfloor \end{cases}, \tag{2.19}$$

and $\lambda_N' := \lambda_N(1-\varepsilon)$ where $1 > \varepsilon > 0$ is as small as we like. In view of the Agmon-Allegretto-Piepenbrink criterion, it is sufficient to prove that the function ψ is superharmonic with parameter $\alpha = -\frac{\lambda_N'}{2}$. To see this, for $|x| < \lfloor N^\Theta \rfloor$ we write

$$\widetilde{H}(N)\psi(x) = \left(\frac{1}{2}\Delta_N + \frac{\lambda_N^2\omega^2}{2}\frac{x^2}{N^2}\right)\psi(x) = e^{-\lambda_N'\omega\frac{x^2}{2N^2}}\left[N^2\left(1-\frac{e^{-\frac{\lambda_N'\omega}{2N^2}}}{2}\left(e^{\lambda_N'\omega\frac{x}{N^2}}+e^{-\lambda_N'\omega\frac{x}{N^2}}\right)\right)+\frac{\lambda_N^2\omega^2}{2}\frac{x^2}{N^2}\right]. \tag{2.20}$$

Since for $|x| < \lfloor N^\Theta \rfloor$,

$$\frac{\lambda_N' x}{N^2} \leq N^{-\delta} \to 0, \quad \text{as } N \to +\infty,$$

we can expand the term inside the square brackets according to a Taylor series, and obtain

$$\left[N^2\left(1-\frac{e^{-\frac{\lambda_N'\omega}{2N^2}}}{2}\left(e^{\lambda_N'\omega\frac{x}{N^2}}+e^{-\lambda_N'\omega\frac{x}{N^2}}\right)\right)+\frac{\lambda_N^2\omega^2}{2}\frac{x^2}{N^2}\right] = \tag{2.21}$$

$$\frac{\lambda_N'\omega}{2} + \omega^2(\lambda_N^2 - \lambda_N'^2)\frac{x^2}{2N^2} - \frac{\lambda_N'^4\omega^4x^4}{24N^6} + \frac{\lambda_N'^3\omega^3x^2}{4N^4} + O(\frac{\lambda_N^6\omega^6x^6}{N^{10}}) \tag{2.22}$$

For N large enough and $|x| < \lfloor N^\Theta \rfloor$,

$$\omega^2(\lambda_N^2 - \lambda_N'^2)\frac{x^2}{2N^2} - \frac{\lambda_N'^4\omega^4x^4}{24N^6} \geq \omega^2(1-(1-\varepsilon)^2)N^{2-\delta}/2 - \omega^4N^{2-4\delta}/24 \geq 0, \tag{2.23}$$

so that, for this regime, (2.20) and (2.21) imply the inequality

$$(\frac{1}{2}\Delta_N + \frac{\lambda_N^2\omega^2}{2}\frac{x^2}{N^2})\psi(x) \geq \psi(x)\frac{\lambda_N'\omega}{2}, \tag{2.24}$$

whenever N is sufficiently large.

If $|x| \geq \lfloor N^\Theta \rfloor + 1$, then $\Delta_N\psi(x) = 0$ and hence

$$(\frac{N^2}{2}\Delta_N + \frac{\lambda_N^2\omega^2}{2}\frac{x^4}{N^2})\psi(x) \geq e^{-\lambda_N'\omega\frac{\lfloor N^\Theta\rfloor^2}{2N^2}}\frac{\lambda_N^2\omega^2}{2}N^{4+2\gamma-4\delta} \geq \psi(x)\frac{\lambda_N'\omega}{2}, \tag{2.25}$$

since $4+2\gamma-4\delta > 2$, which implies that $\omega^2\lambda_N N^{2+4\gamma/3} \geq \lambda'_N\omega$, for N sufficiently large.

At the interface $x = \lfloor N^\Theta \rfloor$, we have

$$\begin{aligned}(\frac{N^2}{2}\Delta_N + \frac{\lambda_N^2\omega^2}{2}\frac{x^2}{N^2})\psi(\lfloor N^\Theta \rfloor) &= e^{-\lambda'_N\omega\frac{\lfloor N^\Theta \rfloor^2}{2N^2}}[N^2(1-\frac{1}{2}(1+e^{-\frac{\lambda'_N\omega}{2N^2}}e^{\lambda'_N\omega\frac{\lfloor N^\Theta \rfloor}{N^2}})) + \lambda_N^2\omega^2\frac{\lfloor N^\Theta \rfloor^4}{2N^2}] \\ &\geq \psi(\lfloor N^\Theta \rfloor)(N^{4+2\gamma-4\delta} - CN^2) \geq \psi(\lfloor N^\Theta \rfloor)\frac{\lambda'_N\omega}{2}, \end{aligned} \tag{2.26}$$

where the last estimate has been performed by observing that

$$(1-\frac{1}{2}(1+e^{-\frac{\lambda'_N\omega}{2N^2}}e^{\lambda'_N\omega\frac{\lfloor N^\Theta \rfloor}{N^2}})) \geq -C,$$

for some positive $C > 0$ and

$$(N^{4+2\gamma-4\delta} - CN^2)/\lambda_N \xrightarrow{N\to+\infty} +\infty.$$

In this way, we discover that ψ is a super solution with parameter $-\lambda'_N\omega/2$. Hence,

$$\liminf_{N\to\infty}\frac{\widetilde{E}_0(N)}{\lambda_N} \geq \frac{\omega(1-\varepsilon)}{2} \tag{2.27}$$

for the energy of the Hamiltonian with $\widetilde{V}_N$ as a potential. Since now ε is arbitrary, we get

$$\liminf_{N\to\infty}\frac{\widetilde{E}_0(N)}{\lambda_N} \geq \frac{\omega}{2}, \tag{2.28}$$

as desired. □

REMARK 2.7: We note that the choice $\lambda'_N = (1-\varepsilon)\lambda_N$ in (2.19) is necessary to guarantee the positivity of Eq. (2.23). We could have also taken $\varepsilon \to \varepsilon_N := N^{-2\delta}$, which would have given the stronger result, for N large but fixed

$$\frac{\widetilde{E}_0(N)}{\lambda_N} \geq \frac{1}{2}\omega + O(\frac{\lambda_N}{N^{2\delta}}). \tag{2.29}$$

◇

Estimate for the first excited state

In the following proofs, we require an estimate for the energy of the first excited state of $\widetilde{H}(N)$. While the Agmon–Allegretto–Piepenbrink criterion is effective for deriving lower bounds, it applies only to the ground state and cannot be used directly in this case. Nevertheless, we can still make use of the criterion by considering a suitably restricted Hamiltonian. The strategy involves several key ideas, which we present in the form of a series of lemmas.

LEMMA 2.8: (**Behavior of eigenfunctions close to zeros**) Let Ω be an eigenfunction with eigenvalue E for an generic discrete Hamiltonian $H = \Delta/2 + V$, then

a) Ω cannot be contemporaneously zero at two subsequent points.

b) If $\Omega(\bar{x}) = 0$, $\Omega(\bar{x}-1) = -\Omega(\bar{x}+1) \neq 0$.

$\diamond$

Proof. a) If by absurd $\Omega(\bar{x}) = \Omega(\bar{x}+1) = 0$, then, since Ω solves the Schrödinger equation, we would have

$$0 = E\Omega(\bar{x}) = 2\Omega(\bar{x}) - \Omega(\bar{x}-1) - \Omega(\bar{x}+1) + V(\bar{x})\Omega(\bar{x}) = -\Omega(\bar{x}-1), \tag{2.30}$$

so that $\Omega(\bar{x}-1) = 0$. By iteration, we discover that Ω is identically zero, which is absurd.

b) If $\Omega(\bar{x}) = 0$, exploiting again the eigenvalue equation we get that

$$\Omega(\bar{x}+1) + \Omega(\bar{x}-1) = 0, \tag{2.31}$$

and since $\Omega(x \pm 1) \neq 0$ by a), the two values of the eigenfunction must have opposite sign and the same modulus. $\square$

The next lemma characterizes the behavior of the zeros of higher energy eigenstates.

LEMMA 2.9: **(Zeros of excited states)** Suppose we are given a discrete Hamiltonian $H := \Delta/2 + V$ and call its eigenvectors by Ω_m. If the nth eigenfunction Ω_n changes sign k times, then Ω_{n+1} changes sign at least $k+1$ times.

$\diamond$

Proof. Let us take $x_0 < x_1 \in \mathbb{Z}$ such that Ω_n is positive in $[x_0, x_1]$ and $\Omega_n(x_0-1)$, $\Omega_n(x_1+1) < 0$. Suppose by absurd that also Ω_{n+1} doesn't change sign and, w.l.o.g., that it is positive in $[x_0, x_1]$. Then we can compute

$$\frac{1}{2}\sum_{x=x_0}^{x_1} \left(\Omega_n(x)\Delta\Omega_{n+1}(x) - \Omega_{n+1}(x)\Delta\Omega_n(x)\right) = (E_1 - E_0)\sum_{x=x_0}^{x_1} \Omega_n(x)\Omega_{n+1}(x) > 0, \tag{2.32}$$

where the equality is a consequence of the discrete Schrödinger equation for Ω_{n+1}, Ω_n. We can also compute the sum explicitely by expanding the laplacians

$$\begin{aligned}
&\sum_{x=x_0}^{x_1} \big(\Omega_n(x)\Delta\Omega_{n+1}(x) - \Omega_{n+1}(x)\Delta\Omega_n(x)\big) \\
&= \sum_{x=x_0}^{x_1} \big(\Omega_n(x+1)\Omega_{n+1}(x) + \Omega_n(x-1)\Omega_{n+1}(x) - \Omega_{n+1}(x+1)\Omega_n(x) - \Omega_{n+1}(x-1)\Omega_n(x)\big) \\
&= \Omega_n(x_1+1)\Omega_{n+1}(x_1) + \Omega_n(x_0-1)\Omega_{n+1}(x_0) - \Omega_{n+1}(x_1+1)\Omega_n(x_1) - \Omega_{n+1}(x_0-1)\Omega_n(x_0). \quad (2.33)
\end{aligned}$$

If $\Omega_{n+1}(x_1+1)$ and $\Omega_{n+1}(x_0-1)$ were positive or zero, then, all four term in the last line of (2.33) would be negative or zero, leading to a contradiction with the result in (2.32). So, either $\Omega_{n+1}(x_1+1)$ or $\Omega_{n+1}(x_0-1) < 0$.

If Ω_n were positive (or negative) at $x_0 = x_1$ and negative (positive) outside, then, by the same argument, the only possibility for Ω_{n+1} is that $\Omega_{n+1}(x_0) = 0$

Moreover, the same proof works if $x_0 = -\infty$ or $x_1 = +\infty$. So, starting from $-\infty$, we discover that Ω_{n+1} changes signs or is zero inside every interval where Ω_n is either strictly positive or strictly negative. So, the number of times Ω_{n+1} changes signs is equal to the number of intervals where Ω_n has a definite sign, that is $k+1$.

□

REMARK 2.10: Clearly the proof extend to the case in which Ω_n has infinite zeros. In that case, the previous argument shows that also Ω_{n+1} has infinite zeros.

◇

We are now in a position to prove that the first excited state of $\widetilde{H}(N)$ is odd, with only one zero at $x=0$. Indeed, we can consider two new Hamiltonians $\widetilde{H}_r(N)$ and $\widetilde{H}_\ell(N)$, defined respectively on $\mathbb{Z}_+ :\ = [1,+\infty)\cap\mathbb{Z}$ and $\mathbb{Z}_- :\ = (-\infty,-1]\cap\mathbb{Z}$:

$$\widetilde{H}_\ell(N) :\ = \frac{1}{2}(\Delta_N)_{|\mathbb{Z}_-} + \lambda_N^2 \widetilde{V}_N + \frac{N^2}{2}\delta_{x,-1}, \tag{2.34}$$

$$\widetilde{H}_r(N) :\ = \frac{1}{2}(\Delta_N)_{|\mathbb{Z}_+} + \lambda_N^2 \widetilde{V}_N + \frac{N^2}{2}\delta_{x,1}, \tag{2.35}$$

where

$$\begin{aligned}(\Delta_N)_{|\mathbb{Z}_+} g(1) &:\ = g(1) - g(2),\\ (\Delta_N)_{|\mathbb{Z}_+} g(x) &:\ = 2g(x) - g(x-1) - g(x+1), \quad x > 1,\end{aligned} \tag{2.36}$$

and similarly for $(\Delta_N)_{|\mathbb{Z}_-}$. Here, $\delta_{x,y}$ denotes the Kronecker delta.

If $f\colon \mathbb{Z}\to\mathbb{C}$ is such that $f(0) = 0$, it is easy to verify that

$$\widetilde{H}_\ell(N) f_{|\mathbb{Z}_-}(x) = \widetilde{H}(N) f(x), \tag{2.37}$$

$$\widetilde{H}_r(N) f_{|\mathbb{Z}_+}(x) = \widetilde{H}(N) f(x). \tag{2.38}$$

At this point, one can, at least in principle, solve the eigenvalue equations for $\widetilde{H}_\ell(N)$ and $\widetilde{H}_r(N)$ and find the strictly positive ground states $\widetilde{\Omega}_{\ell,0}$, $\widetilde{\Omega}_{r,0}$ for the two Hamiltonians. By simple symmetry reasoning, it is clear that $\Omega_{\ell,0}(x,N) = \Omega_{r,0}(-x,N)$ and that their eigenvalues must coincide, that is, with obvious notation, $\widetilde{E}_\ell(N) = \widetilde{E}_r(N) := \widetilde{E}(N)$. In the remaining of the proof we omit the second index N in the definition of $\Omega_{\ell,0}$ and $\Omega_{r,0}$.

Coming back to the original problem, we define the following function on $\mathbb{Z}$

$$\widetilde{\psi}_1(x) :\ = \begin{cases} \frac{1}{\sqrt{2}}\Omega_{\ell,0}(x) & x < 0 \\ 0 & x = 0\,. \\ -\frac{1}{\sqrt{2}}\Omega_{r,0}(x) & x > 0 \end{cases} \tag{2.39}$$

It is clear from (2.37), (2.38) and from the definition of $\widetilde{\psi}_1$ that

$$\begin{aligned}
&\widetilde{H}(N)\widetilde{\psi}_1(x) = \widetilde{H}_\ell(N)\frac{1}{\sqrt{2}}\widetilde{\Omega}_{\ell,0}(x) = \widetilde{E}(N)\widetilde{\Omega}_{r,0}(x) \quad x \le -1,\\
&\widetilde{H}(N)\widetilde{\psi}_1(x) = \widetilde{H}_r(N)\frac{1}{\sqrt{2}}\widetilde{\Omega}_{r,0}(x) = \widetilde{E}(N)\widetilde{\Omega}_{r,0}(x) \quad x \ge 1,\\
&\widetilde{H}(N)\widetilde{\psi}_1(0) = 0 = \widetilde{\psi}_1(0)\widetilde{E}(N)
\end{aligned} \tag{2.40}$$

so that $\widetilde{\psi}_1$ is an eigenvector of $\widetilde{H}(N)$. Moreover, $\widetilde{\psi}_1$ has only one zero at $x = 0$, so, by lemma 2.9 it necessarily coincides with the first excited state $\widetilde{\Omega}_1$ with energy $\widetilde{E}(N) = \widetilde{E}_1(N)$. We have proved the following:

PROPOSITION 2.11: Let $\widetilde{H}(N) = \Delta_N/2 + \lambda_N^2\widetilde{V}_N$ be the modified Hamiltonian with potential (2.17). Then, the first excited state is anti-symmetric and has only one zero at $x = 0$.

$\diamond$

We proceed now to estimate from below the value of $\widetilde{E}_1$. To do this, we consider the Hamiltonian $\widetilde{H}_r(N)$ (equivalently, we could have also taken $\widetilde{H}_\ell(N)$) and we employ the Agmon-Allegretto-Pieperbrink criterion with super solution

$$\phi(x) := \begin{cases} xe^{-\lambda_N'\omega\frac{x^2}{2N^2}} & \text{if } 1 \le x < \lfloor N^\Theta \rfloor \\ \lfloor N^\Theta \rfloor e^{-\lambda_N'\omega\frac{\lfloor N^\Theta \rfloor^2}{2N^2}} & x \ge \lfloor N^\Theta \rfloor \end{cases} \tag{2.41}$$

and parameter $\alpha = -\frac{3}{2}\lambda_N'\omega$, where as before, $\lambda_N' = \lambda_N(1-\varepsilon)$, for $\varepsilon > 0$ arbitrary small.

At this stage, it is sufficient to repeat the computations of the previous paragraph to verify that $\widetilde{H}_r(N)\phi(x) \ge \frac{3}{2}\omega\lambda_N'\phi(x) + \phi(x)o(\lambda_N)$.

For $x > \lfloor N^\Theta \rfloor$ ϕ is constant, so, we only have the potential term

$$\widetilde{H}_r(N)\phi(x) = \phi(x)\frac{\lambda_N^2 x^4}{2N^2} \ge \phi(x)CN^{4+2\gamma-4\delta} \ge \frac{3}{2}\omega\lambda_N'\phi(x), \tag{2.42}$$

for N large enough.

At the interface $x = \lfloor N^\Theta \rfloor$ we have

$$\begin{aligned}
&\widetilde{H}(N)\phi(\lfloor N^\Theta \rfloor)\\
=&\phi(\lfloor N^\Theta \rfloor)\big[N^2(1-\frac{1}{2}(1+e^{-\omega\frac{\lambda_N'}{2N^2}}e^{\omega\lambda_N'\frac{\lfloor N^\Theta \rfloor}{N^2}}(1-\frac{1}{\lfloor N^\Theta \rfloor}))) + \omega^2\frac{\lambda_N^2\lfloor N^\Theta \rfloor^4}{2N^2}\big]\\
\ge&\phi(\lfloor N^\Theta \rfloor)(\frac{\omega^2}{2}N^{4+2\gamma-4\delta} - CN^2) \ge \phi(\lfloor N^\Theta \rfloor)\frac{3}{2}\omega\lambda_N'.
\end{aligned} \tag{2.43}$$

For $1 \le x < \lfloor N^\Theta \rfloor$, we get the following equation

$$\widetilde{H}(N)\phi(x) = \phi(x)\big[N^2(1+\frac{1}{2}e^{-\frac{\omega\lambda_N'}{2N^2}}(e^{-\frac{\lambda_N\omega x}{N^2}}(1+\frac{1}{x}) + (e^{\frac{\lambda_N\omega x}{N^2}}(1-\frac{1}{x}))) + \frac{\lambda_N^2\omega^2x^2}{2N^2}\big]. \tag{2.44}$$

We focus on the coefficient multiplying $\phi(x)$ and we expand in Taylor series for $x\lambda'_N/N^2 \xrightarrow{N\to+\infty} 0$

$$
\begin{aligned}
&N^2(1+\frac{1}{2}e^{-\frac{\omega\lambda'_N}{2N^2}}(e^{-\frac{\lambda_N\omega x}{N^2}}(1+\frac{1}{x})+(e^{\frac{\lambda_N\omega x}{N^2}}(1-\frac{1}{x})))+\frac{\lambda_N^2\omega^2x^2}{2N^2}\\
=&N^2(1-\frac{1}{2}(1-\frac{\omega\lambda'_N}{2N^2}+\frac{\omega^2\lambda_N'^2}{8N^4}+O(\frac{\lambda_N'^3}{N^6}))(2-2\frac{\omega\lambda'_N}{N^2}+\frac{\omega^2\lambda_N'^2x^2}{N^4}-\frac{\omega^3\lambda_N'^3x^2}{3N^6}+O(\frac{x^4\lambda_N^4}{N^8})))+\frac{\omega^2\lambda_N^2x^2}{2N^2}\\
=&\frac{3}{2}\omega\lambda'_N+\frac{\omega^2(\lambda_N^2-\lambda_N'^2)x^2}{2N^2}+\frac{5\omega^3\lambda_N'^3x^2}{12N^4}-O(\frac{x^4\lambda_N^4}{N^6}+\frac{\lambda_N'^2}{N^2}).
\end{aligned}
\tag{2.45}
$$

The previous computations establish that

$$
\widetilde{E}_1(N)\geq\frac{3}{2}\omega\lambda'_N+o(\lambda'_N) \tag{2.46}
$$

for N large enough. Thus, we can conclude that

$$
\liminf_{N\to\infty}\frac{\widetilde{E}_1(N)}{\lambda_N}\geq\frac{3}{2}\omega(1-\varepsilon), \tag{2.47}
$$

and since ε is now arbitrary, we can send it to zero. Putting all together we have proved

PROPOSITION 2.12: Given the modified Hamiltonian $\widetilde{H}(N)$ with potential specified by (2.17), the first excited energy satisfy the asymptotic lower bound

$$
\liminf_{N\to\infty}\frac{\widetilde{E}_1(N)}{\lambda_N}\geq\frac{3}{2}\omega. \tag{2.48}
$$

◇

Generalization for higher order eigenstates In this paragraph, we will prove the following theorem

THEOREM 2.13: Let $\widetilde{H}_0(N)$ be the Hamiltonian for the modified harmonic oscillator with potential given by (2.17). Then, it holds

$$
\liminf_{N\to+\infty}\frac{\widetilde{E}_n(N)}{\lambda_N}\geq\omega(n+\frac{1}{2}) \tag{2.49}
$$

◇

The main idea behind the subsequent proofs is that if we are able to restrict the eigenvalue problem to some subgraphs, call them $\{\mathbb{Z}^n_{k,k+1}\}_{k=0}^n$, where the excited states can be taken strictly positive, the problem will be transformed into one of finding the correct ground state for some restricted Hamiltonians. We will employ supersolutions consisting in a suitable discretization of the harmonic oscillator eigenstates. As will be clear from the proof of Th. 2.13, the union of the subgraphs $\{\mathbb{Z}^n_{k,k+1}\}_{k=0}^n$ cannot be equal to the whole graph, since we have to remove the edges between points where the harmonic oscillator's solution is zero. The following discussion motivates the correct choice for the points we will remove. For a continuous harmonic oscillator, normalized as follows

$$
H^{cont}(N)=-\frac{N^2}{2}\Delta+\frac{\lambda_N'^2\omega^2}{2}x^2, \tag{2.50}
$$

we know that the nth eigenfunction is given, apart from a normalization factor, by

$$\Omega_n^{cont}(x,N) = H_n(x\frac{\sqrt{\lambda_N \omega}}{N})e^{-\frac{\lambda_N \omega x^2}{2N^2}}, \quad x \in \mathbb{R}. \tag{2.51}$$

We consider the discretization of these special functions

$$\mathbb{Z} \ni x \to H_n(x\frac{\sqrt{\lambda_N' \omega}}{N})e^{-\frac{\lambda_N' \omega x^2}{2N^2}}. \tag{2.52}$$

Now, we denote by $\{\bar{x}_k^n\}_{k=1}^n$, the n zeros of the continuous hermite functions $\mathbb{R} \ni x \to H_n(x\sqrt{\omega\lambda_N'}/N)$, ordered as $\bar{x}_k^n < \bar{x}_{k+1}^n$. Moreover, we define $\alpha_N^0 := \sqrt{\omega\lambda_N'}/N$. It is clear that in general $\bar{x}_k^n \notin \mathbb{Z}$, however, we know that $H_n(\lfloor \bar{x}_k^n \rfloor \alpha_N^0)$ and $H_n((\lfloor \bar{x}_k^n \rfloor + 1)\alpha_N^0)$ have opposite signs. Suppose we remove the points $\{\lfloor \bar{x}_k^n \rfloor\}_{k=1}^n$ from $\mathbb{Z}$ and consider as an example the subgraph $[\lfloor \bar{x}_k^n \rfloor + 1, \lfloor \bar{x}_{k+1}^n \rfloor - 1] \cap \mathbb{Z}$. Suppose w.l.o.g. that $H_n(x\alpha_N^0) > 0$ on $[\lfloor \bar{x}_k^n \rfloor + 1, \lfloor \bar{x}_{k+1}^n \rfloor]$. Now, let us test a Laplacian with Dirichlet boundary conditon with the tentative supersolution

$$\varphi_k(x) := H_n(x\alpha_N^0)e^{-\frac{(\alpha_N^0 x)^2}{2}}, \quad x \in \mathbb{Z} \tag{2.53}$$

restricted to $[\lfloor \bar{x}_k^n \rfloor + 1, \lfloor \bar{x}_{k+1}^n \rfloor - 1] \cap \mathbb{Z}$. Then, for $x = \lfloor \bar{x}_{k+1}^n \rfloor - 1$ we have

$$\begin{aligned}\Delta\varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor - 1) &= \varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor - 1) - \frac{1}{2}\varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor - 2) \\ &\geq \varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor - 1) - \frac{1}{2}\Big[\varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor - 2) + \varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor)\Big],\end{aligned} \tag{2.54}$$

where the last inequality uses that $\varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor) \geq 0$. Inequality (2.54) allows to recover the correct symmetry for the problem, which we had originally broken by imposing the Dirichlet boundary conditions. However, at $x = \lfloor x_k^n \rfloor + 1$ we have instead

$$\Delta\varphi_k(\lfloor \bar{x}_k^n \rfloor + 1) = \varphi_k(\lfloor \bar{x}_k^n \rfloor + 1) - \frac{1}{2}\varphi_k(\lfloor \bar{x}_{k+1}^n \rfloor + 2), \tag{2.55}$$

and we cannot recover the original Laplacian on $\mathbb{Z}$,since $\varphi(\lfloor \bar{x}_k^n \rfloor)/2$ is negative and it cannot be added to (2.55) if we search for a lower bound.

The correct way to proceed is to modify the supersolution (2.53) by rescaling its argument. We take $\alpha_N^k := (\bar{x}_k^n / \lfloor \bar{x}_k^n \rfloor)\sqrt{\lambda_N' \omega}/N$ and

$$\phi_k(x) := H_n(x\alpha_N^k)e^{-\frac{(\alpha_N^k x)^2}{2}}, \quad x \in \mathbb{Z}. \tag{2.56}$$

Then, $\phi_k(\lfloor \bar{x}_k^n \rfloor) = 0$ and

$$\Delta\phi_k(\lfloor \bar{x}_k^n \rfloor + 1) = \phi_k(\lfloor \bar{x}_k^n \rfloor + 1) - \frac{1}{2}\Big[\phi_k(\lfloor \bar{x}_{k+1}^n \rfloor + 2) + \phi_k(\lfloor \bar{x}_{k+1}^n \rfloor)\Big], \tag{2.57}$$

so that the correct Laplacian is recovered. Now,depending on the value of $\bar{x}_k^n / \lfloor \bar{x}_k^n \rfloor$, we could have $\phi_k(\lfloor \bar{x}_k^n \rfloor) \leq 0$, however, this problem ca be solved by simply shifting the right boundary of $[\lfloor \bar{x}_k^n \rfloor + 1, \lfloor \bar{x}_{k+1}^n \rfloor - 1]$ to the left. The correct way of choosing the correct supersolutions and the subgraphs is illustrated in the following definition:

DEFINITION 2.14: (i) We denote by $\{\bar{x}_k^n\}_{k=1}^n$, the ordered n zeros of the continuous hermite functions $\mathbb{R} \ni x \to H_n(x\sqrt{\omega\lambda_N'}/N)$. Moreover, we call $\alpha_N^0 := \sqrt{\omega\lambda_N'}/N$.

(ii) If $n = 1$, we take

$$x_0^1 := -\infty,\ x_1^1 := 0,\ x_2^1 := +\infty, \quad \mathbb{Z}_{0,1}^n := \mathbb{Z} \cap (x_0^n, x_1^n),\ \mathbb{Z}_{1,2} := \mathbb{Z} \cap (x_1^n, x_2^n) \tag{2.58}$$

(iii) For $n > 1$, we define a set of $n+2$ points, and $n+1$ subgraphs of $\mathbb{Z}$ as follows:

$$x_0^n = -\infty,\ x_1^n := \lfloor \bar{x}_1^n \rfloor, \quad \mathbb{Z}_0^n := \mathbb{Z} \cap (-\infty, x_1^n), \tag{2.59}$$

so that $H_n(x_1^n \alpha_N^0)$ and $H_n((x_1^n + 1)\alpha_N^0)$ have the same sign. Then, for $\bar{x}_2^n$, we proceed by setting $\alpha_N^1 := (\bar{x}_1^n / x_1^n)\sqrt{\lambda_N'\omega}/N$ and

$$x_2^n := \lfloor \bar{x}_2^n \rfloor - \ell_2, \tag{2.60}$$

where $\ell_2 \in \mathbb{N}$[1] is the smallest natural number such that

$$(\lfloor \bar{x}_2^n \rfloor - \ell_2)\frac{\bar{x}_1^n}{x_1^n} \leq \bar{x}_2^n. \tag{2.61}$$

Then, we define the subgraph $\mathbb{Z}_{1,2}^n := \mathbb{Z} \cap (x_1^n, x_2^n)$. With these choices, we obtain that $H_n(x_1^n \alpha_N^1) = 0$; moreover $H_n((x_2^n - 1)\alpha_N^1)$ and $H_n(x_2^n \alpha_N^1)$ have the same sign. We continue with this construction: if $\bar{x}_k^n \neq 0$ we define $\alpha_N^k := (\bar{x}_k^n / x_k^n)\sqrt{\lambda_N'\omega}/N$; instead if $\bar{x}_k^n = 0$, $\alpha_N^k := \sqrt{\lambda_N'\omega}/N$. Then we set $\mathbb{Z}_{k,k+1}^n := \mathbb{Z} \cap (x_k^n, x_{k+1}^n)$, where x_{k+1}^n is defined as above, to ensure that $H_n(x_k^n \alpha_N^k) = 0$ and $H_n(x_{k+1}^n \alpha_N^k)$ and $H_n((x_{k+1}^n - 1)\alpha_N^k)$ have the same sign.

◇

Proof of Th. 2.13 . Consider the nth energy level $\widetilde{E}_n(N)$ and eigenstate $\widetilde{\Omega}_n$. Consider now the points $\{\bar{x}_k^n\}_{k=1}^n$. All these points depend on N, and as $N \to +\infty$, they are of order $O(\sqrt{\lambda_N'}/N)$ (the same is true for the points $\{x_k^n\}_{k=1}^n$).

Then, we define on $\ell^2(\mathbb{Z}_{k,k+1}^n)$ the Hamiltonian

$$\widetilde{H}_{k,k+1}(N) := \frac{1}{2}\Delta_N|_{\mathbb{Z}_{k,k+1}^n} + \lambda_N^2 \widetilde{V}_N + \frac{N^2}{2}\delta_{x_k^n+1} + \frac{N^2}{2}\delta_{x_{k+1}^n-1}, \tag{2.62}$$

where the Laplacian $\Delta_N|_{\mathbb{Z}_{k,k+1}^n}$ is taken with Dirichlet boundary conditions and we define $\delta_{\pm\infty} := 0$. Clearly, if $f \in \ell^2(\mathbb{Z})$ is such that $f(x_k) = f(x_k + 1) = 0$, then

$$\widetilde{H}(N)f(x) = \widetilde{H}_{k,k+1}^n(N) f|_{\mathbb{Z}_{k,k+1}^n}(x), \quad x \in \mathbb{Z}_{k,k+1}^n. \tag{2.63}$$

For every Hamiltonian $\widetilde{H}_{k,k+1}(N)$, there exists a strictly positive and non degenerate ground state defined in $\mathbb{Z}_{k,k+1}$. We will denote these ground states as Ω_n^k, $k \in \{1, \dots, n\}$ and their energies as $\widetilde{E}_n^k$. The idea

[1] note that, with this definition, ℓ_k can always be bounded from above by some integer independent from N.

is that we can glue these solutions into one tentative eigenfunction $\widetilde{\psi}_n$. The latter will not satisfy an eigenvalue equation for $\widetilde{H}(N)$ because it will have different energies for different regions of $\mathbb{Z}$. However, we plan to solve the opposite problem, that is, we will find the correct Hamiltonian for $\widetilde{\psi}_n$ and use it to estimate $\widetilde{H}(N)$.

We start by defining our *wannabe* eigenfunction $\widetilde{\psi}_n \in \ell^2(\mathbb{Z})$

$$\widetilde{\psi}_n(x) := \begin{cases} a_n^k \Omega_n^k(x,N) & x \in \mathbb{Z}_{k,k+1}^n \\ 0 & x \in \{x_1^n, \dots, x_n^n\} \end{cases}, \tag{2.64}$$

where the coefficients a_k^n are chosen so to satisfy

$$a_n^k \Omega_n^k(x_k^n, N) + a_n^{k+1} \Omega_n^{k+1}(x_k^n + 1, N) = 0 \quad \textbf{(boundary condition at zeros)}, \tag{2.65}$$

$$\sum_{k=0}^{n} \left[(a_n^k)^2 \right] = 1 \quad \textbf{(normalization condition)}. \tag{2.66}$$

Let us now take the smallest of the ground state energies

$$\widetilde{E}_n(N) := \min\{\widetilde{E}_n^0, \dots, \widetilde{E}_n^n\}. \tag{2.67}$$

This value can be estimated by using the Agmon-Allegretto-Piepenbrink criterion 2.5 applied for every $\widetilde{H}_{k,k+1}(N)$ in $\mathbb{Z}_{k,k+1}^n$, using $n+1$ tentative supersolution. The computation of this estimate for every region will be carried on in Prop. A.1, for every k

$$\widetilde{E}_n^k \geq (\frac{1}{2} + n)\lambda_N' \omega + O(\frac{\lambda_N'^{3/2}}{N}). \tag{2.68}$$

We already know how to obtain an upper bound in a generic region

$$\widetilde{E}_n^k \leq (\frac{1}{2} + n)\lambda_N \omega + O(\frac{\lambda_N'^{3/2}}{N}), \tag{2.69}$$

by standard estimates for ground states energies *cf.* Sec. 2.3, Prop. 2.1 and Rmk. 2.3. Bounds (2.68) and (2.69) finally imply that

$$\widetilde{E}_n(N) - \widetilde{E}_n^k \in -\varepsilon(\frac{1}{2} + n)\omega\lambda_N + O(\frac{\lambda_N'^{3/2}}{N}), \quad \text{for } k \in \{0, \dots, n\}. \tag{2.70}$$

To account for the difference between the energies (2.70), we define another Hamiltonian $\widehat{H}(N)$ as follow:

$$\widehat{H}(N) := \widetilde{H}(N) + \sum_{k=0}^{n} (\widetilde{E}_n(N) - \widetilde{E}_n^k)\chi_{\mathbb{Z}_{k,k+1}^n}, \tag{2.71}$$

where $\{\chi_{\mathbb{Z}_{k,k+1}^n}\}_{k=0}^n$ are characteristic functions of $\mathbb{Z}_{k,k+1}^n$, interpreted as multiplication operators. Notice that, by definition

$$0 \geq -\varepsilon(n + \frac{1}{2})\lambda_N \omega + O(\frac{\lambda_N'^{3/2}}{N}) \geq \widetilde{E}_n(N) - \widetilde{E}_n^k, \tag{2.72}$$

so that

$$\widetilde{H}(N) \geq \widehat{H}(N), \tag{2.73}$$

and

$$\widehat{H}(N) \geq \widetilde{E}_0(N) + \sum_{k=0}^{n} (\widetilde{E}_n(N) - \widetilde{E}_n^k)\chi_{\mathbb{Z}^n_{k,k+1}} \geq \frac{\lambda_N \omega}{2}[(1-\varepsilon) - \varepsilon(n+\frac{1}{2})] + O(\frac{\lambda_N'^{3/2}}{N}) > 0, \tag{2.74}$$

The first inequality in Eq. (2.74) uses the estimate for the ground state energy derived in Prop. 2.6, while the last inequality holds for N large enough and ε sufficiently small. Now, Inequality (2.73) implies [19, XIII, Prob. 1]

$$\widetilde{E}_n(N) \geq \widehat{E}_n(N), \tag{2.75}$$

where $\widehat{E}_n(N)$ is the nth eigenvalue for $\widehat{H}(N)$.

One can now easily verify that

$$\begin{aligned} &\widehat{H}(N)\widetilde{\psi}_n(x) = \widetilde{H}(N)\widetilde{\psi}_n|_{\mathbb{Z}^n_{k,k+1}}(x) \\ &+(\widetilde{E}_n(N) - \widetilde{E}_n^k)\widetilde{\psi}_n(x) = \left[\widetilde{E}_n^k + (\widetilde{E}_n(N) - \widetilde{E}_n^k)\right]\widetilde{\psi}_n(x) = \widetilde{E}_n(N)\widetilde{\psi}_n(x), \quad x \in \mathbb{Z}^n_{k,k+1} && (2.76) \\ &\widehat{H}(N)\widetilde{\psi}_n(x_k^n) = 0, && (2.77) \end{aligned}$$

so that, $\widetilde{\psi}_n$ is an eigenfunction of $\widehat{H}(N)$ with eigenvalue $\widetilde{E}_n(N)$. Now, $\widetilde{\psi}_n$ has n changes of sign, so that by lemma 2.9, it has to be the nth eigenfunction of $\widehat{H}(N)$[2]. Putting together this last observation and Eq. (2.75) with (2.68), we obtain at last

$$\widetilde{E}_n(N) \geq (\frac{1}{2}+n)\lambda_N'\omega + O(\frac{\lambda_N'^{3/2}}{N}). \tag{2.78}$$

□

Oscillators on $\mathbb{Z}^d$ Now, we have at our disposal upper bound estimates for the asymptotic value of the energies of a one dimensional harmonic oscillator $H^{harm}(N)$, and similar estimates for the lower bound of the energy levels of a one dimensional modified oscillator $\widetilde{H}(N)$. These results are sufficient to obtain estimates for the d-dimensional oscillators on $\mathbb{Z}^d$. Indeed, the oscillators are by definition decoupled along the different directions, e.g.

$$H^{harm,d}(N)\psi(x) = \frac{N^2}{2}\sum_{i=1}^{d}\sum_{j_i=-1}^{1}(\psi(x) - \psi(x_1,\dots x_i + j_i,\dots x_d)) + \frac{\lambda_N^2}{2N^2}\sum_{i=1}^{d}\omega_i^2 x_i^2\psi(x). \tag{2.79}$$

Then, it is clear that if we order as in (1.11), the elements of the sets

$$\Big\{\sum_{\alpha=1}^{d}\omega_\alpha(m_\alpha + \frac{1}{2}),\, m_1,\dots,m_d \in \mathbb{N}\Big\}, \tag{2.80}$$

[2]Note that, by (2.74), $\widehat{H}(N)$ is strictly positive for N large enough. Thus, its ground state has to be strictly positive and so, this statement is justified by an inductive argument.

we have

$$\limsup_{N\to+\infty} \frac{E_n(N)}{\lambda_N} \le e_n(V^{harm}), \tag{2.81}$$

$$\liminf_{N\to+\infty} \frac{\widetilde{E}_n(N)}{\lambda_N} \ge e_n(V^{harm}), \tag{2.82}$$

where $E_n(N)$ and $\widetilde{E}_n(N)$ are the nth energy levels of the harmonic and modified d-dimensional oscillators.

2.3 Upper bound for the energy levels of a generic potential

In this section we will estimate the asymptotic values of the *low-lying* eigenvalues for the Hamiltonian

$$H(N) = \frac{1}{2}\Delta_N + \lambda_N^2 V_N \quad \text{on } \ell^2(\mathbb{Z}^d), \tag{2.83}$$

where $V_N(x) = V(x/N)$ and V satisfies the Assumptions 1. Profiting by the preparatory results for the harmonic oscillator, *cf.* Prop. 2.1, we have an extended version for $H(N)$.

PROPOSITION 2.15:

Let $H(N)$ as above and $\gamma \in (-1,1)$, then for N large enough $H(N)$ has at least m eigenvalues,denoted by $\{E_n(N)\}_{n=0}^{m-1}$, below its continuous spectrum. Moreover, the following upper bound holds true

$$\limsup_{N\to+\infty} \frac{E_n(N)}{\lambda_N} \le e_n(V), \tag{2.84}$$

where $e_n(V)$ has been defined in (1.11).

◇

Proof. We proceed in a similar way as the proof of Prop. 2.1. Consider the test functions

$$\psi_{i,n_i}(x) := \frac{1}{C_N^{(n_i)}} \prod_{\alpha=1}^{d} J_N(x_\alpha - Na_{i,\alpha}) H_{n_\alpha}(x_\alpha \frac{\sqrt{\lambda_N \omega_\alpha(a_i)}}{N}) e^{-\frac{\lambda_N \ \omega_\alpha(a_i)(x_\alpha - Na_{i,\alpha})^2}{2N^2}}, \tag{2.85}$$

where the values $\omega_\alpha(a_i)$ have been defined shortly before Eq. (1.11), the indices n_i and $\{n_\alpha\}_{\alpha=1}^d$ labeling the test function and the Hermite polynomials are such that

$$\sum_{\alpha=1}^{d} \omega_\alpha(a_i)(n_\alpha + \frac{1}{2}) = e_{n_i}(V). \tag{2.86}$$

Take N large enough to guarantee

$$\lambda_N > N^{2\delta} (\min_{i,j} \|a_i - a_j\|_\infty)^{-2}, \tag{2.87}$$

so that the test functions have disjoint supports.

$$\langle \psi_{i,n_i}, \psi_{j,n_j} \rangle = \langle \psi_{i,n_i}, H(N)\psi_{j,n_j} \rangle = 0, \quad i \neq j. \tag{2.88}$$

Moreover, it is also easy to verify that

$$\lim_{N\to+\infty} \langle \psi_{i,n_i}, \psi_{i,m_i} \rangle = 0, \quad \text{for } n_i \neq m_i \tag{2.89}$$

$$\lim_{N\to+\infty} \langle \psi_{i,n_i}, \frac{H(N)}{\lambda_N}\psi_{i,m_i} \rangle = 0, \quad \text{for } n_i \neq m_i. \tag{2.90}$$

Let us denote by $H_i(N)$ the harmonic oscillator $H^{harm}(N)$ concentrated in a_i with frequencies $\omega_\alpha(a_i)$, *cf.* Eq. 1.11. Then, we can estimate

$$\begin{aligned}
\langle \psi_{i,n}, H(N)\psi_{i,n} \rangle &= \langle \psi_{i,n}, H_i(N)\psi_{i,n_i} \rangle + \langle \psi_{i,n_i}, H(N) - H_i(N)\psi_{i,n_i} \rangle \leq e_{n_i}(V) + O(N^{1/2-3\gamma/2}) \\
&+2 \sum_{x_d=\lfloor Na_{i,d}-\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor}^{\lfloor Na_{i,d}+\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor} \cdots \sum_{x_1=\lfloor Na_{i,1}-\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor}^{\lfloor Na_{i,1}+\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor} \psi_{i,n_i}(x)^2 \lambda_N^2 (V_N(x) - V_N^{harm}(x)) \\
&\leq e_{n_i}(V) + \frac{2\lambda_N^2}{(C_N^{(n)})^2} \sum_{x_d=\lfloor Na_{i,d}-\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor}^{\lfloor Na_{i,d}+\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor} \cdots \sum_{x_1=\lfloor Na_{i,1}-\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor}^{\lfloor Na_{i,1}+\frac{N^{1+\delta}}{\lambda_N^{1/2}}\rfloor} \prod_{\alpha=1}^{d} e^{-\lambda_N \omega_\alpha(a_i)\frac{(x_\alpha - Na_{i,\alpha})^2}{N^2}} R(\frac{x-Na_i}{N}),
\end{aligned} \tag{2.91}$$

where R is the remainder in V's expansion. The last term is of order

$$\frac{\lambda_N^{5/2}}{N} \sum_{x=0}^{\frac{N^{1+\delta}}{\lambda^{1/2}}} e^{-\lambda_N \frac{x^2}{N^2}} \frac{x^3}{N^3} \leq C \frac{\lambda_N^{5/2}}{\lambda_N^{3/2}} \frac{1}{\lambda_N^{1/2}} + o(\lambda^{1/2}) = O(\lambda^{1/2}). \tag{2.92}$$

Then, we get the estimate

$$\limsup_{N\to\infty} \frac{\langle \psi_{i,n_i}, H(N)\psi_{i,n_i} \rangle}{\lambda_N} \leq e_{n_i}(V), \quad 1 \leq i \leq \kappa. \tag{2.93}$$

Now, since the infimum of the essential spectrum satisfies $\inf\{\sigma_{\text{ess}}(H(N))\} \geq C\lambda_N^2$, for N large enough there will be at least n discrete eigenvalues and we can use equations (2.88),(2.89),(2.90) and (2.93) together with the Rayleigh-Riesz principle [19, Th. XIII.3] to find

$$\limsup_{N\to\infty} \frac{E_n(N)}{\lambda_N} \leq e_n(V), \quad 1 \leq i \leq \kappa. \tag{2.94}$$

$\square$

2.4 Lower bound for the energy levels of a generic potential

In this section, we are interested in obtaining an optima lower bound for $H(N)$'s eigenvalues. It will be necessary to employ the *IMS localization formula* [17] and define a refined test function. The proof is similar in spirit to [17, Th. 3.2], with some major technical differences due to the non locality of the Laplacian and the discreteness of the problem.

PROPOSITION 2.16: Let the potential satisfy Assumptions 1 with $\gamma \in (-1,1)$. Then, the following lower bound holds true

$$\liminf_{N\to\infty} \frac{E_n(N)}{\lambda_N} \geq e_n(V), \quad \text{for all } n \geq 0$$

◇

Proof. We take a bump function $k : \mathbb{R} \to \mathbb{R}$ which is C^∞, equal to one for $|x| \leq 1$ and 0 for $|x| \geq 2$ and $0 \leq k(x) \leq 1$ for all $x \in \mathbb{R}$. Then we define

$$K_i(x) = \prod_{\alpha=1}^{d} K_{i,\alpha}(x) := \prod_{\alpha=1}^{d} k\left(\frac{\lambda_N^{1/2}}{N^\delta}(x_\alpha/N - a_{i,\alpha})\right), \quad 1 \leq i \leq \kappa, \tag{2.95}$$

where we take $\delta > 0$, satisfying $\gamma - 6\delta > -1$ and $1 > \gamma + \delta$. The latter condition is necessary so that we can take N large enough to satisfy at least

$$\lambda_N > 16N^{2\delta}(\min_{i,j}|a_i - a_j|)^{-2}, \tag{2.96}$$

so that the bumb functions all have disjoint supports centered around the minima of the potential. We also define $K_0^2 := 1 - \sum_{i=1}^{\kappa} K_i^2$. We can easily prove a discrete version of the IMS localization formula. By identifying K_i with the respective multiplication operators we get

$$K_i^2 H(N) + H(N)K_i^2 - 2K_i H(N) K_i = [K_i, [K_i, H(N)]]. \tag{2.97}$$

This latter formula implies

$$H(N) = \sum_{i=0}^{k} K_i H(N) K_i + \frac{1}{2}\sum_{i=0}^{k} [K_i, [K_i, H_N]] \tag{2.98}$$

To obtain a good estimate for $H(N)$ it is necessary to evaluate the norm of the double commutators in (2.98). This is possible on $\ell^2(\mathbb{Z}^d)$ because the Laplacian is a bounded operator[3] Now, the bumb function K_i is defined as the product of *one dimensional functions* $K_{i,\alpha}$, and since the Laplacian is a sum of operators acting along the different spatial directions of the lattice, we can obtain an estimate by simply studying the one dimensional case. Thus, given an arbitrary $\psi \in \ell^2(\mathbb{Z})$, by an explicit expansion of the commutators we can write

$$[K_{i,\alpha}, H(N)]\psi(x) = [K_{i,\alpha}, \frac{1}{2}\Delta_{\alpha,N}]\psi(x) = -\frac{N^2}{2}\sum_{x\sim y} \psi(y)(K_{i,\alpha}(x) - K_{i,\alpha}(y)). \tag{2.99}$$

[3]The unbound-ness of the continuum Laplacian is recovered in the limit $N \to +\infty$ thanks to the N^2 in front of the discrete one.

$$[K_{i,\alpha},[K_{i,\alpha},H(N)]]\psi(x) = -\frac{N^2}{2}\sum_{x\sim y}\psi(y)(K_{i,\alpha}(x)-K_{i,\alpha}(y))^2. \tag{2.100}$$

Since $[K_{i,\alpha},[K_{i,\alpha},H(N)]]$ is self-adjoint, we can estimate its norm by evaluating

$$\begin{aligned}|\langle\psi,[K_{i,\alpha},[K_{i,\alpha},H(N)]]\psi\rangle| \le \frac{N^2}{2}\sum_{x\in\mathbb{Z}}[&|\psi(x)\psi(x+1)|(K_{i,\alpha}(x)-K_{i,\alpha}(x+1))^2\\ &+|\psi(x)\psi(x-1)|(K_{i,\alpha}(x)-K_{i,\alpha}(x-1))^2].\end{aligned} \tag{2.101}$$

Now, we want to estimate the difference $|K_{i,\alpha}(x)-K_{i,\alpha}(x+1)|$. The latter can be written explicitly as

$$|k(\frac{\lambda_N^{1/2}}{N^{1+\delta}}x - a_i\frac{\lambda_N^{1/2}}{N^\delta}) - k(\frac{\lambda_N^{1/2}}{N^{1+\delta}}x + \frac{\lambda_N^{1/2}}{N^{1+\delta}} - a_i\frac{\lambda_N^{1/2}}{N^\delta})|.$$

By definition of $\mathbb{R}\ni x\to k(x)$ we have the following

- if $x > \lfloor Na_i + 2N^{1+\delta}/\lambda_N^{1/2}\rfloor$, $|K_{i,\alpha}(x)-K_{i,\alpha}(x+1)| = 0$ since k is constant.
- if $Na_i \le x < \lfloor N^{1+\delta}/\lambda_N^{1/2}\rfloor + Na_i$, $|K_{i,\alpha}(x)-K_{i,\alpha}(x+1)| = 0$ since k is constant.
- if $x-\lfloor Na_i\rfloor \in [\lfloor N^{1+\delta}/\lambda_N^{1/2}\rfloor,\ \lfloor 2N^{1+\delta}/\lambda_N^{1/2}\rfloor]$,
$$|K_{i,\alpha}(x)-K_{i,\alpha}(x+1)| \le \sup_{x\in\mathbb{R}}|k'(x)|\lambda_N^{1/2}/N^{1+\delta} \le C\lambda_N^{1/2}/N^{1+\delta}.$$

By symmetry for $x\to Na_i - x$, we can obtain the same estimates for $x < Na_i$, for all $i\in\{1,\dots,\kappa\}$. In this way we get

$$|\langle\psi,[K_{i,\alpha},[K_{i,\alpha},H(N)]]\psi\rangle| \le C^2\frac{\lambda_N}{N^{2\delta}}\sum_{x\in\mathbb{Z}}|\psi(x)|(|\psi_+(x)|+|\psi_-(x)|) \le 2C^2\frac{\lambda_N}{N^{2\delta}}\|\psi\|^2, \tag{2.102}$$

where we have defined

$$\psi_+(x) := \psi(x+1), \quad \psi_-(x) := \psi(x-1), \tag{2.103}$$

and in the last step we have used Cauchy-Schwarz inequality. Finally we get $\|[K_i,[K_i,H(N)]]\| \in O(\lambda_N/N^\delta)$. The same computations lead to $\|[K_0,[K_0,H_N]]\| \in O(\lambda_N/N^{2\delta})$.

Now we are in a position to estimate $E_n(N)/\lambda_N$ by looking directly at the Hamiltonian and comparing it with the modified oscillator Hamiltonian

$$\widetilde{H}_i(N) := \frac{1}{2}\Delta_N + \lambda_N^2\widetilde{V}_N(x-a_i), \tag{2.104}$$

with frequencies chosen as $\omega_\alpha \equiv \omega_\alpha(a_i)$. Moreover, the δ appearing in the definition of $\widetilde{V}_N$, *cf.* (2.17), is take to be the same as the one we have inserted in Eq. (2.95). By making use of the IMS localization formula (2.98), we write the Hamiltonian as

$$H(N) = K_0H(N)K_0 + \sum_{i=1}^{k}K_i\widetilde{H}_i(N)K_i + \sum_{i=1}^{k}K_i(H(N)-\widetilde{H}_i(N))K_i + \sum_{i=0}^{k}[K_i,[K_i,H(N)]] \tag{2.105}$$

For

$$x \notin \bigcup_{i=1}^{\kappa}[Na_i - \frac{N^{1+\delta}}{\lambda_N^{1/2}}, Na_i + \frac{N^{1+\delta}}{\lambda_N^{1/2}}]$$

we have, that

$$\lambda_N^2 V_N(x) \geq \min_{1\leq i\leq \kappa} \lambda_N^2 V_N\big(Na_i + \frac{N^{1+\delta}}{\lambda_N^{1/2}}\big) = C\lambda_N N^{2\delta} + O(\lambda_N^{1/2} N^{3\delta}), \tag{2.106}$$

where the first inequality uses that V_N has a local minimum at Na_i, $i \in \{1,\ldots,\kappa\}$, while for the second inequality we have Taylor expanded the potential around one of these minima. Since $K_0\Delta K_0$ is a positive operator and $\delta > 0$, it follows that for large N

$$K_0 H(N) K_0 \geq K_0^2 C\lambda_N N^{2\delta} \geq \lambda_N e_0(d) K_0^2 \tag{2.107}$$

For the second term, using the lower bound derived in Th 2.13, we simply have

$$K_i \widetilde{H}_i(N) K_i \geq K_i^2 \frac{1}{2}\sum_{\alpha=1}^{d} \omega_\alpha(a_i) + O(\frac{\lambda_N^{3/2}}{N}) \geq K_i^2 e_0(d) + O(\frac{\lambda_N^{3/2}}{N}). \tag{2.108}$$

Now we examine the operators $K_i(H(N) - \widetilde{H}_i(N))K_i$. The support of the cut-off functions K_i is centered around $x = Na_i$ and it is of order $N^{1+\delta}/\lambda_N^{1/2}$, while the modified potential $\widetilde{V}_N$ differs from the harmonic one for $x \geq \lfloor N^{1+\gamma-\delta} \rfloor$. With our hypothesis on δ and γ we have

$$1 + \gamma - \delta > \frac{1}{2} + \delta + \frac{\gamma}{2} \tag{2.109}$$

so that the non-harmonic part disappears in the following estimate

$$K_i(H(N) - \widetilde{H}_i(N))K_i = \lambda_N^2 K_i^2 \Big(V_N(x) - \widetilde{V_N}(x - Na_i)\Big) = \lambda_N^2 K_i^2 \Big(V_N(x) - \sum_{\alpha=1}^{d} \frac{\omega_\alpha(a_i)(x_\alpha - Na_i)^2}{2N^2}\Big). \tag{2.110}$$

By Taylor expanding V_N around Na_i, we discover that this last term is of order $O(\lambda_N^{1/2} N^{3\delta})$. The double commutants have already be estimated to be of order $O(\lambda_N/N^{2\delta})$. Inserting all the due estimates in the Eq. (2.98), we find

$$H(N) \geq \sum_{i=0}^{k} K_i^2 e_0(d)\lambda_N + O(\frac{\lambda_N^{3/2}}{N} + \frac{\lambda_N}{N^{2\delta}} + \lambda_N^{1/2} N^{3\delta}). \tag{2.111}$$

To conclude, we get

$$\liminf_{N\to+\infty} \frac{E_0(N)}{\lambda_N} \geq e_0(d). \tag{2.112}$$

Now, we argue by induction to estimate the excited levels. We proceed as follows: suppose we have proven that

$$\liminf_{N\to+\infty} \frac{E_n(N)}{\lambda_N} \geq e_n(V) \tag{2.113}$$

and suppose that $e_{n+1}(V) > e_n(V)$ (if they were equal the estimate would be trivial since $E_{n+1}(N) \geq E_n(N)$). Then, we take P_i to be the projection on the eigenfunctions of $\widetilde{H}_i(N)$ having energy less than $e_{n+1}(V)\lambda_N + O(\lambda_N^{3/2}/N)$. Again, we can estimate

$$K_0 H(N) K_0 \geq K_0^2 C \lambda_N N^{2\delta} \geq e_{n+1}(V)\lambda_N K_0^2. \tag{2.114}$$

Then, we can bound the Hamiltonian $\widetilde{H}_i$, for arbitrary i, from below as follow

$$K_i\widetilde{H}_iK_i = K_iP_i\widetilde{H}_iP_iK_i + K_i\widetilde{H}_i(1-P_i)K_i \geq K_iP_i(\widetilde{H}_i - Ie_{n+1}(V)\lambda_N)P_iK_i + e_{n+1}(V)\lambda_N K_i^2 + O(\frac{\lambda_N^{3/2}}{N}), \tag{2.115}$$

where we have estimated $K_i\widetilde{H}_i(1-P_i)K_i \geq e_{n+1}(V)\lambda_N K_i(1-P_i)K_i + O(\lambda_N^{3/2}/N)$. Subsequently, the whole Hamiltonian can be estimated as follow

$$H(N) \geq e_{n+1}(V)\lambda_N I + \sum_{i=1}^{k} K_iP_i(\widetilde{H}_i(N) - Ie_{n+1}(V)\lambda_N)P_iK_i - O(\frac{\lambda_N}{N^{2\delta}}) + O(\frac{\lambda_N^{3/2}}{N}) \tag{2.116}$$

Since $\sum_{i=1}^{k} K_iP_i(\widetilde{H}_i(N) - Ie_{n+1}(V)\lambda_N)P_iK_i :\, = F_n$ has rank at most n, we can take an arbitrary vector $\varphi \in \mathrm{Ran}(F_n)^{\perp}$, $\|\varphi\| = 1$ and find

$$\langle \varphi, H(N)\varphi \rangle \geq e_{n+1}(V)\lambda_N - O(\frac{\lambda_N}{N^{2\delta}}) - O(\frac{\lambda_N^{3/2}}{N}). \tag{2.117}$$

However, by the min-maximum principle, the $(n+1)$th level $E_{n+1}(N)$ satisfies

$$E_{n+1}(N) = \sup_{\varphi_1,\dots\varphi_k} \inf_{\varphi \in [\varphi_1,\dots,\varphi_k]^{\perp}} \langle \varphi, H(N)\varphi \rangle \geq \inf_{\varphi \in \mathrm{Ran}(F_n)^{\perp}} \langle \varphi, H(N)\varphi \rangle. \tag{2.118}$$

Then, thanks to (2.117) we conclude that

$$E_{n+1}(N) \geq e_{n+1}(V)\lambda_N - O(\frac{\lambda_N}{N^{2\delta}}) - O(\frac{\lambda_N^{3/2}}{N}), \tag{2.119}$$

and lastly

$$\liminf_{N\to\infty} \frac{E_{n+1}(N)}{\lambda_N} \geq e_{n+1}(V), \tag{2.120}$$

concluding the proof of the theorem. □

In summary, putting together Prop. 2.15 and Prop. 2.16 we have obtained the existence of the limit

$$\lim_{N\to+\infty} \frac{E_n(N)}{\lambda_N} = e_n(V), \tag{2.121}$$

that is, we have proved Th. 1.2.

3 Other regions of the parameter space

In this section we briefly discuss qualitatively what happens if we take into consideration other values of the scaling parameter γ. In particular, we can individuate five relevant regions for the latter parameter.

Continuum limit of the free discrete Laplacian $\gamma > 1$: In this limit λ_N decreases, when $N \to +\infty$, as $N^{-(\gamma-1)}$. We expect that all the possible discrete eigenvalues of $H(N)$ converge to 0, leaving only a residual essential spectrum; that is, the correct limit of Eq. (1.14) is in some sense the continuum Laplacian, with spectrum given by the whole $[0,+\infty)$. This is confirmed by Figure 1 below, for the regime $(+\infty, 1)$. Indeed, when the eigenvalues converge to zero, the logarithm tends to $-\infty$. To make this more precise, we have the following proposition.

PROPOSITION 3.1: Let $H(N)$ satisfy Assumptions 1 with $\gamma > 1$. Assume also that the potential V in the definition of $H(N)$ grows for $|x| \to +\infty$ at most as $e^{|x|^m}$, $m > 0$. Then we have

$$\lim_{N\to+\infty} E_n(N) = 0, \quad \text{for all } n \geq 0. \tag{3.1}$$

$\diamond$

Proof. We will give a proof only for $d = 1$, the extension to all $d > 1$ being trivial but more lengthy. We exploit again the Rayleigh-Ritz principle [19, Th XIII.3] to obtain an upper bound for all the eigenvalues. We can construct a suitable basis of approximately orthonormal test function as follows: define

$$\beta_N := 1 + \frac{1}{m\log(N)}\Big(\log((2\gamma-2-\varepsilon)\log(N))\Big) > 1 \tag{3.2}$$

for some $\varepsilon > 0$, $2\gamma - 2 - \varepsilon > 0$, and take

$$g_0(x) := \begin{cases} 2 + \dfrac{x}{\lfloor N^{\beta_N}\rfloor} & 2\lfloor N^{\beta_N}\rfloor < x < -\lfloor N^{\beta_N}\rfloor \\ 1 & |x| \leq \lfloor N^{\beta_N}\rfloor \\ 2 - \dfrac{x}{\lfloor N^{\beta_N}\rfloor} & \lfloor N^{\beta_N}\rfloor < x < 2\lfloor N^{\beta_N}\rfloor \\ 0 & |x| \geq 2\lfloor N^{\beta_N}\rfloor \end{cases}. \tag{3.3}$$

Then, we proceed to construct g_n by inserting n-nodes in a symmetric fashion between $(-\lfloor N^{\beta_N}\rfloor, \lfloor N^{\beta_N}\rfloor)$, e.g. for $n = 1$

$$g_1(x) = \begin{cases} 1 & -\lfloor N^{\beta_N}\rfloor < x < -\lfloor N^{\beta_N}/4\rfloor \\ -\dfrac{x}{\lfloor N^{\beta_N}/4\rfloor} & -\lfloor N^{\beta_N}/4\rfloor \leq x \leq \lfloor N^{\beta_N}/4\rfloor \\ -1 & \lfloor N^{\beta_N}/4\rfloor < x < -\lfloor N^{\beta_N}\rfloor \end{cases}. \tag{3.4}$$

At this point one can easily see that

$$\lim_{N\to+\infty} \langle \frac{g_n}{\|g_n\|}, \frac{g_m}{\|g_m\|} \rangle = 0, \quad \text{for } n \neq m. \tag{3.5}$$

Moreover, we can estimate for the Laplacian

$$\frac{\langle g_n, N^2 \Delta g_n \rangle}{\langle g_n, g_n \rangle} \in O(N^{2-2\beta_N}). \tag{3.6}$$

For the potential term we have

$$\frac{\langle g_n, \lambda_N^2 V_N g_n \rangle}{\langle g_n, g_n \rangle} \leq C \max_{x \in [-2\lfloor N^{\beta_N} \rfloor, 2\lfloor N^{\beta_N} \rfloor] \cap \mathbb{Z}} \frac{V(x/N)}{N^{2\gamma-2}} \leq C' \frac{e^{N^{m(\beta_N-1)}}}{N^{2\gamma-2}}. \tag{3.7}$$

With our choice for β_N then, we have

$$\begin{aligned} \frac{e^{N^{m(\beta_N-1)}}}{N^{2\gamma-2}} &= N^{-\varepsilon} \xrightarrow{N\to+\infty} 0, \\ N^{2(1-\beta_N)} &= e^{2(1-\beta_N)\log(N)} \leq C'' e^{-\log(\log(N))} \xrightarrow{N\to+\infty} 0, \end{aligned}$$

and these estimate are sufficient to prove that

$$\lim_{N\to+\infty} E_n(N) = 0. \tag{3.8}$$

□

Purely continuum limit $\gamma = 1$: For $\gamma = 1$, the prefactor in front of the potential is $\lambda_N = 1$ and the discrete Hamiltonian $H(N)$ becomes the discrete approximation of the continuum Hamiltonian

$$H := -\frac{1}{2}\Delta + V. \tag{3.9}$$

As we have explicitly shown for the harmonic oscillator, see Prop. 2.1,2.13, we expect that also the discrete eigenvalues of a generic $H(N)$ converge, without any rescaling, to the energies of the corresponding continuum model. This interpretation is further supported by the observations in [6], where it is explicitly stated that for (1.4), "the limit $\delta \to 0$ for fixed $\lambda > 0$ corresponds to the problem of the continuum limit, and various quantities associated with $H_\delta(\lambda)$ converge to those of $H_{\text{cont}}(\lambda)$." This perspective is also consistent with earlier discussions in [5, 7]. In view of Figure 1, the convergence of eigenvalues implies that the ratio $\frac{\log E_n^{harm}(N)}{\log N}$ tends to zero, confirming these observations.

Semiclassical approximation $1 > \gamma > -1$: We have examined this region in the previous sections, showing how the combined continuum and semiclassical limit of $H(N)$ allows us to precisely approximate the semi-classical eigenvalue asymptotics of $H_S^{cont}(\lambda)$, the latter encoded by the semi-classical limit of κ harmonic oscillators in the continuum, whose frequencies are given by the square root of the eigenvalues of the Hessians $(\partial_{x_\alpha}\partial_{x_\beta}V(a_i))_{\alpha,\beta}$. In our second paper on this topic, we conduct a detailed analysis of this interval, showing that a genuine semiclassical limit emerges, not only in terms of eigenvalue asymptotics, but also for eigenvector asymptotics. In particular, we recover the same Agmon estimates as those obtained by Simon in [15]. For further details, we refer the reader to the discussion that follows.

Purely discrete model $\gamma = -1$: A preliminary analysis shows how the behavior of the Hamiltonian $H(N)$ and of its eigenvalue becomes really model-dependent for $\gamma = -1$. Indeed, if for example we focus on the simplest case of the one dimensional harmonic oscillator, we see that

$$(H^{harm}(N)f)(x) = N^2\left(\frac{1}{2}\sum_{|x-y|=1}(f(x)-f(y)) + \frac{1}{2}\omega^2x^2f(x)\right) \quad \text{on} \quad \ell^2(\mathbb{Z}), \tag{3.10}$$

that is, the parameter N is factorized in front of everything, and the model retains it discrete nature for every N, with the only difference being that the eigenvalues of the discrete harmonic oscillator now scale as N^2.

The break down of the continuum approximation could have been deduced also from our proofs for the upper and lower bounds of the eigenvalues of the harmonic oscillator, see Prop. 2.1,2.6, A.1. In all the cases, the test functions we implemented contained the exponential factor $e^{-\lambda_N x^2/N^2}$ and the proofs highly depended on the possibility of expanding this latter function for large N, as a mean to recover the continuum Laplacian. However, for $\lambda_N = N^2$, the N-dependence disappears from the argument of the exponential and we are not able to Taylor expand our test functions. As a consequence, locality of Δ is lost and we are not in the continuum limit regime anymore.

Semiclassical approximation for discrete models $\gamma < -1$: It is expected that, in this parameter range, the semiclassical limit is approached more rapidly than the continuum limit. We will analyze some specific models, from which it will be clear that in this case the setting is similar to the one of [12], where the eigenfunctions localize around every point of the potential.

EXAMPLE 3.2: As a first example we consider the discrete one dimensional harmonic oscillator (the d-dimensional case behaves similarly)

$$H^{harm}(N) = \frac{1}{2}\Delta_N + N^{2|\gamma|}\frac{\omega^2x^2}{2} \quad \text{on} \quad \ell^2(\mathbb{Z}/N). \tag{3.11}$$

It is easier to study the Hamiltonian $H^{harm}(N)/N^{2|\gamma|}$, where the only dependence on N left is on the factor $N^{-2|\gamma|}$ in front of the Laplacian. Clearly, if $\psi_f \in \ell^2(\mathbb{Z})$ is of finite support, then $\psi_f \in D(H^{harm}(N)/N^{|2\gamma|})$ for all $N > 1$ and

$$\frac{H^{harm}(N)}{N^{2|\gamma|}}\psi_f \xrightarrow{\|\cdot\|} V^{harm}\psi_f, \tag{3.12}$$

so that $H^{harm}(N)/N^{2|\gamma|}$ converges in the strong-resolvent sense [18, Th. VIII.25] to the multiplication operator V^{harm}. In particular, this implies [18, Th. VIII.24] that the eigenvalues of $H(N)$ satisfy

$$\begin{aligned}
&\lim_{N\to+\infty}\frac{E_0(N)}{N^{2|\gamma|}} = 0,\\
&\lim_{N\to+\infty}\frac{E_{2n-1}}{N^{2|\gamma|}} = \lim_{N\to+\infty}\frac{E_{2n}}{N^{2|\gamma|}} = \frac{\omega^2n^2}{2}, \quad \text{for } n \geq 0.
\end{aligned} \tag{3.13}$$

Note that the eigenvalues do not scale anymore as λ_N, but as λ_N^2/N^2, as confirmed in [12, Sec. 2.1]. This is clear from Figure 1, in the parameter regime $(-1, -\infty)$.

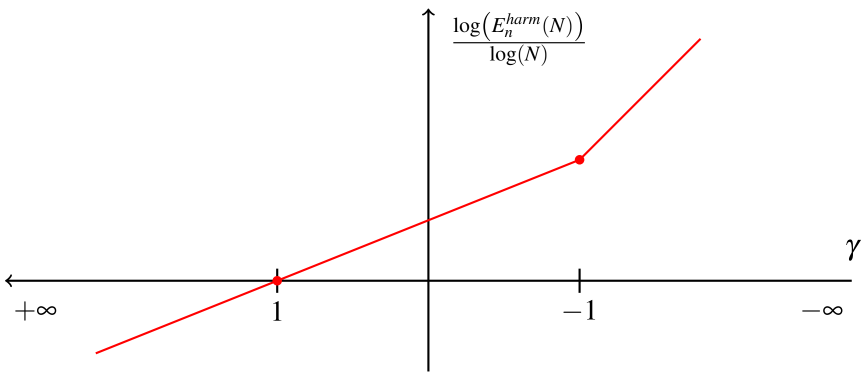

Figure 1: Dependence on γ

◇

EXAMPLE 3.3: Suppose that in addition to Assumptions 1 $V \in \mathbb{L}^\infty(\mathbb{R}^d)$, then V_N, as a multiplication operator is bounded. Clearly, for every $\psi \in \ell^2(\mathbb{Z}^d)$ one has

$$\|V_N\psi - V(0)\psi\|^2 = \sum_{x\in\mathbb{Z}} |\psi(x)|^2 |V(\frac{x}{N}) - V(0)|^2 \xrightarrow{N\to+\infty} 0, \tag{3.14}$$

by a simple dominated convergence argument where $|V(x/N) - V(0)| \leq 2\|V\|_\infty$. Thus, $H(N)/\lambda_N^2$ strongly converges to $V(0)I$, and so[4]

$$\lim_{N\to+\infty} \frac{E_n(N)}{\lambda_N^2} = V(0), \quad \text{for all } n \geq 0. \tag{3.15}$$

This example shows how this setting is not exactly the same as the one of Linn, Lippner and Yau [12], because in their work they only implemented a semiclassical parameter in front of the potential, while the latter was kept fixed. We see that the energies are all degenerate in the limit, and scale as λ_N^2 instead of λ_N.

◇

4 Discussion and future works

In this work, we have studied a suitable scaling limit of a sequence of discrete Schrödinger operators that approximates the semi-classical limit of a continuum Schrödinger operator. Through our analysis, we have demonstrated that the chosen scaling captures the correct asymptotic behavior in the transition from the discrete to the continuum and semi-classical setting. This not only validates the appropriateness of

[4]Depending on the the potential, for fixed N there could be only a finite number of discrete eigenvalues; what we are saying is that for N large enough, the nth eigenvalue exists and its limit is given by Eq. (3.15).

our approach but also provides a framework for bridging discrete models with their continuous counterparts in the semi-classical regime. Our results lay the groundwork for further exploration of discrete-to-continuum limits in the semiclassical regime in more intricate settings, including more complex graphs (see below), variable potentials, and the approximation of the macroscopic limit of mean-field quantum spin systems as a combined semiclassical and continuum limit of Schrödinger operators, an idea that has also been foreshadowed in [21]. We conclude with the following final remarks.

REMARK 4.1:

- Instead of considering the scaled lattice $X = \mathbb{Z}/N$, one may also work with a finite approximation of $\mathbb{Z}$, namely, $X_N = [-N^{1+\varepsilon}, N^{1+\varepsilon}] \cap \mathbb{Z}$ with $\varepsilon > 0$ arbitrarily small, with rescaled lattice X_N/N. Indeed, in this way, the length of the interval
$$|[-N^{1+\varepsilon}, N^{1+\varepsilon}]|/N = 2N^{1+\varepsilon}/N \xrightarrow{N\to+\infty} +\infty,$$
so that one recovers a model on the line in the continuum. If $\varepsilon = 0$, $2N^{1+\varepsilon}/N \xrightarrow{N\to+\infty} 2$, i.e. we do not approximate $\mathbb{R}$, but rather a finite box corresponding to a particle in a box, which is a physically different problem.

- It would be of interest to extend the analysis beyond the integer lattice $\mathbb{Z}^d$ to more general classes of graphs. Conducting a similar study in such broader settings could provide deeper insight into the underlying physical phenomena and potentially unify results across discrete structures. In particular, this direction would serve to generalize and complement the existing works on finite graphs by [12, 20].

$\diamond$

REMARK 4.2: **(Eigenfunction localization)**
We would like to draw the reader's attention to the special case $\gamma = 0$, which has been extensively studied in the literature: for $d = 1$ in [4], for general $d \geq 1$ in [11], and in the framework of semiclassical pseudodifferential operators on the torus in [7]. These works rely on suitable variants of the standard Agmon metric to quantify upper bounds for the exponential decay of eigenfunctions. In contrast, in our second work [9], the choice of $\gamma \in (-1,1)$ recovers the classical Agmon metric, as introduced by Agmon in [2]. More precisely, this choice allows us to characterize the localization of eigenfunctions in precisely the same sense as Simon did in the continuum setting [15], thereby establishing the actual existence of the semiclassical limit, rather than merely deriving upper bounds.

$\diamond$

A Appendix

In this Appendix we prove the validity of the lower bound (2.68)for an arbitrary n:

PROPOSITION A.1: Given the set of Hamiltonians $\{\widetilde{H}_{k,k+1}(N)\}_{k=0}^{n}$, defined on the subgraphs $\mathbb{Z}^{n}_{k,k+1}$ as in Eq. (2.62), the lower bound (2.68) holds for every k, i.e.

$$\widetilde{E}^{n}_{k} \geq (n+\frac{1}{2})\lambda'_N\omega + O(\frac{\lambda_N^{3/2}}{N}). \tag{I.1}$$

$\diamond$

Proof. We will exploit some characteristic properties of the Hermite polynomials [1, Ch. 22] (taken with physicists normalization) i.e.

$$-\frac{1}{2}(\partial_x^2 H_n e^{-\frac{x^2}{2}})(x) + \frac{1}{2}x^2 H_n(x)e^{-\frac{x^2}{2}} = (n+\frac{1}{2})H_n(x)e^{-\frac{x^2}{2}} \quad \textbf{(eigenvalue equation)}, \tag{I.2}$$

$$(\partial_x H_n)(x) = 2nH_{n-1}(x), \quad \textbf{(recursion relation)} \tag{I.3}$$

$$H_n(\bar{x}) = 0 \Rightarrow H_m(\bar{x}) \neq 0, \text{ for } m \neq n \quad \textbf{(simplicity of zeros)}. \tag{I.4}$$

To lighten the notation we will use $\Theta := 1+\gamma-\delta$, where $\delta > 0$ was defined in (2.17) so that $\gamma - 2\delta > -1$.

The estimates go as follows. We start by estimating the eigenvalue in $\mathbb{Z}_{0,1} = (-\infty, x_1^n)$. We use a tentative supersolution

$$\phi_0(x) := \begin{cases} |H_n(x\alpha_N^0)|e^{-\frac{(\alpha_N^0 x)^2}{2}} & -\lfloor N^\Theta \rfloor < x \leq x_1^n \\ |H_n(-\lfloor N^\Theta \rfloor \alpha_N^0)|e^{-\frac{(\alpha_N^0 \lfloor N^\Theta \rfloor)^2}{2}} & x \leq -\lfloor N^\Theta \rfloor \end{cases}, \tag{I.5}$$

restricted to $(-\infty, x_1^n)$. We have already defined the constant α_N^0 in 2.14. It is important to recall that, with our choice for x_1^n, $\mathbb{Z} \ni x \to H_n(x\alpha_N^0)$ has always the same sign in $(-\infty, x_1^n]$, so that

$$\begin{aligned}\phi_0(x+1)+\phi_0(x-1) &= |H_n((x+1)\alpha_N^0)e^{-\frac{((x+1)\alpha_N^0)^2}{2}}| + |H_n((x-1)\alpha_N^0)e^{-\frac{((x-1)\alpha_N^0)^2}{2}}| \\ &= |H_n((x+1)\alpha_N^0)e^{-\frac{((x+1)\alpha_N^0)^2}{2}} + H_n((x-1)\alpha_N^0)e^{-\frac{((x-1)\alpha_N^0)^2}{2}}|, \end{aligned} \tag{I.6}$$

for $x \in \mathbb{Z}^n_{0,1}$. To perform precise estimates we need to subdivide the study of $\mathbb{Z}_{0,1}$ in more subregions

$$x < -\lfloor N^\Theta \rfloor, \quad x = -\lfloor N^\Theta \rfloor, \quad -\lfloor N^\Theta \rfloor < x \leq -\frac{N\theta}{\sqrt{\lambda'_N}}, \quad -\frac{N\theta}{\sqrt{\lambda'_N}} < x < x_1^n - 1, \quad x = x_1^n - 1, \tag{I.7}$$

where $\theta > 0$ will be specified later.

For $x < -\lfloor N^\Theta \rfloor$, ϕ_0 is constant, so that

$$\widetilde{H}_{0,1}(N)\phi_0(x) = \frac{\lambda_N^2\omega^2 x^4}{2N^2}\phi_0(x) \geq \frac{N^{2+2\gamma-4\delta}\omega^2}{2} \geq \phi_0(x)\lambda'_N\omega(n+\frac{1}{2}), \tag{I.8}$$

For $x = -\lfloor N^\Theta \rfloor$ we have

$$\widetilde{H}_{0,1}(N)\phi_0(-\lfloor N^\Theta \rfloor) = \phi_0(-\lfloor N^\Theta \rfloor)\Big\{N^2[1-\frac{1}{2}(1+\frac{\phi_0(-\lfloor N^\Theta \rfloor+1)}{\phi_0(-\lfloor N^\Theta \rfloor)})] + \frac{\lambda_N^2 \lfloor N^\Theta \rfloor^4 \omega^2}{2N^2}\Big\}, \tag{I.9}$$

where we have factorized a factor of $\phi_0(-\lfloor N^\Theta \rfloor)$. Since $\mathbb{Z} \ni x \to |H_n(x\alpha_N^0)|$ is a polynomial of order n, which is strictly decreasing for $x < x_1^n \in O(\sqrt{N/\lambda_N})$, and since

$$-\lfloor N^\Theta \rfloor \alpha_N^0 \xrightarrow{N\to+\infty} -\infty, \quad \lfloor N^\Theta \rfloor (\alpha_N^0)^2 \xrightarrow{N\to+\infty} 0, \tag{I.10}$$

the Hermite functions satisfy

$$A_1 \lfloor N^\Theta \rfloor^n \leq |H_n(-\lfloor N^\Theta \alpha_N^0 \rfloor)| \leq A_2 \lfloor N^\Theta \rfloor^n, \tag{I.11}$$

for some positive constants A_1, A_2. Thus, we have

$$\begin{aligned}\frac{\phi_0(-\lfloor N^\Theta \rfloor)}{\phi_0(-\lfloor N^\Theta \rfloor + 1)} =& e^{-\frac{(\alpha_N^0 \lfloor N^\Theta \rfloor)^2}{2}} e^{\frac{(\alpha_N^0 \lfloor N^\Theta + 1 \rfloor)^2}{2}} \frac{|H_n(-\lfloor N^\Theta \rfloor \alpha_N^0)|}{|H_n(-\lfloor N^\Theta + 1 \rfloor \alpha_N^0)} | \\ \leq & C e^{-(\alpha_N^0)^2 \lfloor N^\Theta \rfloor} \frac{|(\lfloor N^\Theta \rfloor)^n|}{|(\lfloor N^\Theta + 1 \rfloor)^n|} \in O(N^0),\end{aligned} \tag{I.12}$$

so that Eq. (I.9) can be estimated as

$$\widetilde{H}_{0,1}(N)\phi_0(-\lfloor N^\Theta \rfloor) \geq \phi_0(-\lfloor N^\Theta \rfloor)\Big(\frac{\lambda_N'^2 \lfloor N^\Theta \rfloor^4 \omega^2}{2N^2} - CN^2\Big) \geq \phi_0(-\lfloor N^\Theta \rfloor)\lambda_N'(n+\frac{1}{2}) \tag{I.13}$$

The last inequality is a consequence of

$$\frac{1}{\lambda_N'}\Big(\frac{\lambda_N'^2 \lfloor N^\Theta \rfloor^4 \omega^2}{2N^2} - CN^2\Big) \geq C' N^{3+3\gamma-4\delta} \xrightarrow{N\to+\infty} +\infty. \tag{I.14}$$

Now, we analyze the region $-\lfloor N^\Theta \rfloor < x < x_1^n - 1$. We recall that ϕ_0 is strictly positive and can be seen as the discretization of a smooth function on $\mathbb{R}$. We define for comodity

$$\phi_0(x) =: \widetilde{\phi}_0(x\alpha_N^0). \tag{I.15}$$

Then, we have

$$\widetilde{H}_{0,1}\phi_0(x) = N^2[\widetilde{\phi}_0(x\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0(x\alpha_N^0 - \alpha_N^0) + \widetilde{\phi}_0(x\alpha_N^0 + \alpha_N^0))] + \widetilde{\phi}_0(x\alpha_N^0)\frac{\lambda_N^2 \omega^2 x^2}{2N^2}. \tag{I.16}$$

Now, we use an exact Taylor expansion with remainder [14, Th. 1.4.1] to write

$$\widetilde{\phi}_0(x\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0(x\alpha_N^0 - \alpha_N^0) + \widetilde{\phi}_0(x\alpha_N^0 + \alpha_N^0)) = -\frac{1}{2}\widetilde{\phi}_0^{(2)}(x\alpha_N^0)(\alpha_N^0)^2 - \frac{(\alpha_N^0)^4}{6}\int_0^1 \mathrm{d}t\, t^3 \widetilde{\phi}_0^{(4)}(x\alpha_N^0 + t\alpha_N^0), \tag{I.17}$$

where the superscripts (2) and (4) denote the order of derivation of $x \to \widetilde{\phi}_0(x)$. Note that, with some abuse of notation, we are regarding $\widetilde{\phi}_0$ as a function of $\mathbb{R}$. Then, using the eigenvalue equation I.2 for

$$\widetilde{\phi}_0(x) = H_n(x)e^{-\frac{x^2}{2}}, \tag{I.18}$$

we have

$$-\frac{N^2}{2}(x\alpha_N^0)^2\widetilde{\phi}_0^{(2)}(x\alpha_N^0) = -(N\alpha_N^0)^2\frac{(x\alpha_N^0)^2}{2}\widetilde{\phi}_0(x\alpha_N^0) + (N\alpha_N^0)^2(n+\frac{1}{2})\widetilde{\phi}_0(x\alpha_N^0)$$
$$= [-\frac{\lambda_N'^2\omega^2x^2}{2N^2} + \lambda_N'(n+\frac{1}{2})]\widetilde{\phi}_0(x\alpha_N). \quad \text{(I.19)}$$

Moreover, since $-\lfloor N^\Theta \rfloor < x < x_1 \in O(\alpha_N^0)$, the dominant piece in $\widetilde{\phi}_0^{(4)}$ is given by the term where all four derivatives are applied to the exponential $e^{-\frac{x^2}{2}}$, so that we can estimate

$$\sup_{t\in(0,1)} |\widetilde{\phi}_0^{(4)}(x\alpha_N^0 + t\alpha_N^0)| \leq A_3(x\alpha_N^0)^{n+4}e^{-\frac{(x\alpha_N^0)^2}{2}} \quad \text{(I.20)}$$

for some positive constant A_3. Now, we want to estimate the absolute value of the nth hermite function $|H_n|$, at points $x\alpha_N^0$.

We consider the region $-\lfloor N^\Theta \rfloor < x \leq -N\theta/\sqrt{\lambda_N'}$, for some constant $\theta > 0$, such that

$$-\theta\frac{N}{\sqrt{\lambda}} < -x_1^n. \quad \text{(I.21)}$$

Then, we have

$$|H_n(x\alpha_N^0)| \geq A_4(x\alpha_N^0)^n, \quad \text{(I.22)}$$

for some positive constant A_4, and if we plug estimates (I.20), (I.21) back to the supersolution equation

$$\widetilde{H}_{0,1}\phi_0(x) = N^2[\widetilde{\phi}_0(x\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0(x\alpha_N^0-\alpha_N^0) + \widetilde{\phi}_0(x\alpha_N^0+\alpha_N^0))] + \widetilde{\phi}_0(x\alpha_N^0)\frac{\lambda_N^2\omega^2x^2}{2N^2}$$
$$= \widetilde{\phi}_0^N(x\alpha_N^0)[\lambda_N'(n+\frac{1}{2}) + (\lambda_N^2-\lambda_N'^2)\omega^2\frac{x^2}{2N^2} - \frac{N^2(\alpha_N^0)^4}{6}\int_0^1 \mathrm{d}t\, t^3\frac{\widetilde{\phi}_0^{(4)}(x\alpha_N^0+t\alpha_N^0)}{\widetilde{\phi}_0(x\alpha_N^0)}]$$
$$\geq \widetilde{\phi}_0^N(x\alpha_N)[\lambda_N'(n+\frac{1}{2}) + (\lambda_N^2-\lambda_N'^2)\omega^2\frac{x^2}{2N^2} - A_5N^2\alpha_N^8x^4]. \quad \text{(I.23)}$$

The last two addenda in (I.23) satisfy

$$(\lambda_N^2-\lambda_N'^2)\omega^2\frac{x^2}{2N^2} - A_5N^2\alpha_N^8x^4 = (\lambda_N^2-\lambda_N'^2)\omega^2\frac{x^2}{2N^2} - A_5\omega^4\frac{\lambda_N'^4x^4}{N^6} \geq 0 \quad \text{(I.24)}$$

for N large enough and $|x| < \lfloor N^\Theta \rfloor$. We conclude that for $-\lfloor N^\Theta \rfloor < x \leq -\theta N/\sqrt{\lambda_N'}$

$$\widetilde{H}_{0,1}\phi_0(x) \geq \lambda_N'(n+\frac{1}{2})\phi_0(x). \quad \text{(I.25)}$$

Now, we focus on $-\theta N/\sqrt{\lambda_N'} < x < x_1^n - 1$. Since $\mathbb{Z} \ni x \to |H_n|(x\alpha_N^0)$ is strictly decreasing, we can estimate $|H_n|(x\alpha_N^0)$ with its value at x_1^n. Recall that $H_n(\bar{x}_1^n\alpha_N^0) = 0$, so that we have

$$|H_n((x_1^n-1)\alpha_N^0)| = |H_n(\bar{x}_1^n\alpha_N^0) + H_n^{(1)}(\bar{x}_1^n\alpha_N^0)\alpha_N^0(\bar{x}_1^n+1-x_1^n) + O((\alpha_N^0)^2)| \geq A_6(1-\{|\bar{x}_1^n|\})\alpha_N^0|H_n^{(1)}(\bar{x}_1^n\alpha_N^0)|, \quad \text{(I.26)}$$

for some constant $A_6 > 0$. Now, thanks to the recursion relation (I.3), it holds $|H_n^{(1)}(\bar{x}_1^n \alpha_N^0)| = 2n|H_{n-1}(\bar{x}_1^n \alpha_N^0)|$. Using property (I.4), we know that every one of the zeros of $x \to H_{n-1}(x\alpha_N^0)$ satisfies

$$|x_k^{n-1} - x_1^n|\alpha_N^0 > B > 0, \tag{I.27}$$

for some N-independent constant B. As above, we can estimate

$$A_6|(1-\{|\bar{x}_1^n|\})H_{n-1}(\bar{x}_1\alpha_N^0)| \geq A_7(\alpha_N^0)^n \bar{x}_1^{n-1} = A_8\alpha_N^0. \tag{I.28}$$

Moreover,

$$\sup_{t\in(0,1)} \widetilde{\phi}_0^{(4)}(x\alpha_N^0 + t\alpha_N^0) \in O(N^0), \quad \text{for } x \in \big(-\theta\frac{N}{\sqrt{\lambda_N'}},\, x_1^n - 1\big). \tag{I.29}$$

Thus, in the interval $-\theta N/\sqrt{\lambda_N'} < x < x_1^n - 1$, we estimate

$$\begin{aligned}
\widetilde{H}_{0,1}\phi_0(x) =& N^2[\widetilde{\phi}_0(x\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0(x\alpha_N^0 - \alpha_N^0) + \widetilde{\phi}_0(x\alpha_N^0 + \alpha_N^0))] + \widetilde{\phi}_0(x\alpha_N^0)\frac{\lambda_N^2\omega^2x^2}{2N^2} \\
=& \widetilde{\phi}_0^N(x\alpha_N^0)[\lambda_N'(n+\frac{1}{2}) + (\lambda_N^2 - \lambda_N'^2)\omega^2\frac{x^2}{2N^2} - \frac{N^2(\alpha_N^0)^4}{6}\int_0^1 \mathrm{d}t\, t^3 \frac{\widetilde{\phi}_0^{(4)}(x\alpha_N^0 + t\alpha_N^0)}{\widetilde{\phi}_0(x\alpha_N^0)}] \\
\geq& \widetilde{\phi}_0^N(x\alpha_N)[\lambda_N'(n+\frac{1}{2}) + (\lambda_N^2 - \lambda_N'^2)\omega^2\frac{x^2}{2N^2} - A_8N^2(\alpha_N^0)^3].
\end{aligned} \tag{I.30}$$

Since $N^2(\alpha_N^0)^3 = \omega^{3/2}\lambda_N^{3/2}/N$, we conclude that

$$\widetilde{H}_{0,1}(N)\phi_0(x) \geq \phi_0(x)\Big(\lambda_N'\omega(n+\frac{1}{2}) + O(\frac{\lambda_N'^{3/2}}{N})\Big). \tag{I.31}$$

It remains to evaluate the equation for $x = x_1^n - 1$. At this point, we have

$$\widetilde{H}_{0,1}\phi_0(x_1^n - 1) = N^2[\widetilde{\phi}_0((x_1^n-1)\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0((x_1^n-1)\alpha_N^0 - \alpha_N^0))] + \widetilde{\phi}_0((x_1^n-1)\alpha_N^0)\frac{\lambda_N^2\omega^2x^2}{2N^2}. \tag{I.32}$$

Now, we know by definition that $\phi_0(x_1^n) \geq 0$, so that we have the lower bound

$$\begin{aligned}
&N^2[\widetilde{\phi}_0((x_1^n-1)\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0((x_1^n-1)\alpha_N^0 - \alpha_N^0))] \\
&\qquad \geq N^2[\widetilde{\phi}_0((x_1^n-1)\alpha_N^0) - \frac{1}{2}(\widetilde{\phi}_0((x_1^n-1)\alpha_N^0 - \alpha_N^0) + \widetilde{\phi}_0((x_1^n-1)\alpha_N^0 + \alpha_N^0))],
\end{aligned} \tag{I.33}$$

and we can proceed to estimate the supersolution equation as in the interval $(-\theta N/\sqrt{\lambda_N'}, x_1^n - 1)$, obtaining

$$\widetilde{H}_{0,1}\phi_0(x_1^n - 1) \geq \phi_0(x_1^n - 1)(\lambda_N'\omega(n+\frac{1}{2}) + O(\lambda_N'^{3/2}/N)) \tag{I.34}$$

We have concluded the estimate on $\mathbb{Z}_{0,1}^n$. We proceed now with the adjacent region $\mathbb{Z}_{1,2}^n$. We consider the tentative super solution obtained by restricting to $\mathbb{Z}_{1,2}^n$ the following

$$\phi_1(x) := \widetilde{\phi}_1(x\alpha_N^1) := |H_n(x\alpha_N^1)|e^{-\frac{\lambda_N'\omega x^2}{2N^2}}, \tag{I.35}$$

where

$$\alpha_N^1 := \frac{\bar{x}_1^n}{x_1^n}\frac{\sqrt{\omega\lambda_N'}}{N} = (1-\frac{\{\bar{x}_1^n\}}{x_1^n})\frac{\sqrt{\omega\lambda_N'}}{N} = \frac{\sqrt{\omega\lambda_N'}}{N} + O(\frac{\lambda_N'}{N^2}). \tag{I.36}$$

For the following, we recall that both x_1^n and x_2^n are of order $O(\sqrt{\lambda_N'}/N)$ and that with our choice of x_2^n, $H_n(x_2^n\alpha_N^1)$ and $H_n((x_2^n-1)\alpha_N^1)$ have the same sign.

We begin with the supersolution equation at x_1^n+1

$$\widetilde{H}_{1,2}(N)\phi_1(x_1^n+1) = N^2[\widetilde{\phi}_1((x_1^n+1)\alpha_N^1) - \frac{1}{2}\widetilde{\phi}_1((x_1^n+2)\alpha_N^1)] + \frac{\lambda_N^2\omega^2(x_1^n+1)^2}{2N^2}\widetilde{\phi}_1((x_1^n+1)\alpha_N^1). \tag{I.37}$$

With our choice of α_N^1, we have

$$\widetilde{\phi}_1(x_1^n\alpha_N^1) = 0, \tag{I.38}$$

so that one can rewrite Eq. (I.37) as

$$\widetilde{H}_{1,2}(N)\phi_1(x_1^n+1) = N^2[\widetilde{\phi}_1((x_1^n+1)\alpha_N^1) - \frac{1}{2}(\widetilde{\phi}_1((x_1^n+2)\alpha_N^1) + \widetilde{\phi}_1(x_1^n\alpha_N^1))] + \frac{\lambda_N^2\omega^2(x_1^n+1)^2}{2N^2}\widetilde{\phi}_1(x\alpha_N^1). \tag{I.39}$$

Using a Taylor expansion with remainder and the eigenvalue equation (I.2) for $\widetilde{\phi}_1$, we rewrite Eq. (I.39) as

$$\begin{aligned}\widetilde{H}_{1,2}(N)\phi_1(x_1^n+1) = \widetilde{\phi}_1((x_1^n+1)\alpha_1^n)\Big[&(N\alpha_N^1)^2(n+\frac{1}{2}) \\ &+ (\lambda_N^2\omega^2 - (\alpha_N^1)^2)\frac{(x_1^n+1)^2}{2N^2} - \frac{N^2(\alpha_N^1)^4}{6}\int_0^1 \mathrm{d}t\, t^3\frac{\widetilde{\phi}_1^{(4)}((x_1^n+1)\alpha_N^1+t\alpha_N^1)}{\widetilde{\phi}_1((x_1^n+1)\alpha_1^n)}\Big].\end{aligned} \tag{I.40}$$

Now, we estimate the different pieces:

$$(N\alpha_1^n)^2(n+\frac{1}{2}) = \Big(\frac{\bar{x}_1^n}{x_1^n}\Big)^2\lambda_N'(n+\frac{1}{2})\omega = \omega\lambda_N'(n+\frac{1}{2}) + O(\frac{\lambda_N'^{3/2}}{N}), \tag{I.41}$$

$$(\lambda_N^2 - \frac{1}{\omega^2}(\alpha_N^1)^2)\frac{\omega^2(x_1^n+1)^2}{2N^2} \geq -B_1\lambda_N'(\Big(\frac{\bar{x}_1^n}{x_1^n}-1\Big)^2) \in O(\frac{\lambda_N'^{3/2}}{N}) \tag{I.42}$$

$$-\frac{N^2(\alpha_N^1)^4}{6}\int_0^1 \mathrm{d}t\, t^3\frac{\widetilde{\phi}_1^{(4)}((x_1^n+1)\alpha_N^1+t\alpha_N^1)}{\widetilde{\phi}_1((x_1^n+1)\alpha_1^n)} \geq -B_2\frac{\lambda_N'^{3/2}}{N}, \tag{I.43}$$

for some positive constat $B_1, B_2 > 0$. Note that in (I.43) we have used that $\widetilde{\phi}_1^{(4)}((x_1^n+1)\alpha_N^1+t\alpha_N^1) \in O(N^0)$, while $\widetilde{\phi}_1((x_1^n+1)\alpha_1^n) \geq B_3\sqrt{\lambda_N'}/N$, for some constant B_3. Using the three estimate (I.41),(I.42),(I.43) in Eq. (I.40) we have

$$\widetilde{H}_{1,2}(N)\phi_1(x_1^n+1) \geq \phi_1(x_1^n+1)(\lambda_N'\omega(n+\frac{1}{2}) - O(\frac{\lambda_N'^{3/2}}{N})). \tag{I.44}$$

For the region in the interior of $\mathbb{Z}^n_{1,2}$, $x^n_1 < x < x^n_2 - 1$, one has

$$\begin{aligned}\widetilde{H}_{1,2}(N)\phi_1(x) =&N^2[\widetilde{\phi}_1(x\alpha^1_N) - \frac{1}{2}(\widetilde{\phi}_1(x\alpha^1_N + \alpha^1_N) + \widetilde{\phi}_1(x\alpha^1_N - \alpha^1_N))] + \frac{\lambda_N^2\omega^2x^2}{2N^2}\widetilde{\phi}_1(x\alpha^1_N)\\ =&\widetilde{\phi}_1(x\alpha^n_1)\Big[(N\alpha^1_N)^2(n+\frac{1}{2}) + (\lambda_N^2\omega^2 - (\alpha^1_N)^2)\frac{x^2}{2N^2} - \frac{N^2(\alpha^1_N)^4}{6}\int_0^1 \mathrm{d}t\, t^3\frac{\widetilde{\phi}_1^{(4)}(x\alpha^1_N + t\alpha^1_N)}{\widetilde{\phi}_1(x\alpha^n_1)}\Big].\end{aligned} \tag{I.45}$$

and since $x \in O(N/\sqrt{\lambda'_N})$, the estimate follows as before by looking at the different terms in (I.45). For the rightmost point of the inerval, $x^n_2 - 1$, we have made sure that $\phi_1(x_2) \geq 0$, so that we have

$$\begin{aligned}\widetilde{H}_{1,2}(N)\phi_1(x^n_2 - 1) =&N^2[\widetilde{\phi}_1((x^n_2-1)\alpha^1_N) - \frac{1}{2}\widetilde{\phi}_1((x^n_2-2)\alpha^1_N)] + \frac{\lambda_N^2\omega^2(x^n_2-1)^2}{2N^2}\widetilde{\phi}_1((x^n_2-1)\alpha^1_N)\\ \geq&N^2[\widetilde{\phi}_1((x^n_2-1)\alpha^1_N) - \frac{1}{2}(\widetilde{\phi}_1((x^n_2-2)\alpha^1_N) + \widetilde{\phi}_1(x^n_2\alpha^1_N))] + \frac{\lambda_N^2\omega^2(x^n_2-1)^2}{2N^2}\widetilde{\phi}_1((x^n_2-1)\alpha^1_N),\end{aligned} \tag{I.46}$$

and again, by Taylor expanding and using the eigenvalue equation , we can estimate (I.46) as

$$\widetilde{H}_{1,2}(N)\phi_1(x^n_2 - 1) \geq \phi_1(x^n_2-1)(\lambda'_N\omega(n+\frac{1}{2}) + O(\frac{\lambda_N'^{3/2}}{N})), \tag{I.47}$$

concluding the proof in $\mathbb{Z}^n_{1,2}$.

For the subsequent step, we define the supersolution

$$\phi_2(x) := \widetilde{\phi}_2(x\alpha^N_2) := |H_n(x\alpha^2_N)|e^{-\frac{\alpha^2_N x^2}{2N^2}}, \tag{I.48}$$

restricted to $\mathbb{Z}_{2,3}$. Again, we will have

$$\phi_2(x^n_2) = 0,\ \phi_2(x^n_3) > 0. \tag{I.49}$$

Moreover, if $x \in \mathbb{Z}^n_{2,3}$, then it is of order $O(N/\sqrt{\lambda'_N})$ and

$$\alpha^N_2 = \omega\frac{\sqrt{\lambda'_N}}{N}(1 + O(\frac{\sqrt{\lambda'_N}}{N})), \tag{I.50}$$

so that all the estimates we made for $\mathbb{Z}^n_{1,2}$ can be repeated without any change. We proceed by induction on k, for extimating the eigenvalue on $\mathbb{Z}^n_{k,k+1}$, $k < n$. Once we have arive at $k = n$, we use the supersolution obtained by restricting to $\mathbb{Z}^n_{n,n+1}$

$$\phi_n(x) := \widetilde{\phi}_n(x\alpha^n_N) := \begin{cases} |H_n(x\alpha^n_N)|e^{-\frac{\alpha^n_N x^2}{2N^2}} & x^n_n < x < \lfloor N^\Theta \rfloor \\ |H_n(\lfloor N^\Theta \rfloor \alpha^n_N)|e^{-\frac{\alpha^n_N \lfloor N^\Theta \rfloor^2}{2N^2}} & x \geq \lfloor N^\Theta \rfloor \end{cases}, \tag{I.51}$$

This time, $\phi_n(x_n^n) = 0$. Then, we have for $x = x_n^n + 1$,

$$\begin{aligned}\widetilde{H}_{n,n+1}(N)\phi_n(x_n^n+1) =& N^2[\phi_n(x_n^n+1) - \frac{1}{2}\phi_n(x_n^n+2)] + \phi_n(x_n^n+1)\frac{\lambda_N^2\omega^2x^2}{2N^2} \\ =& N^2[\phi_n(x_n^n+1) - \frac{1}{2}(\phi_n(x_n^n+2) + \phi_n(x_n^n))] + \phi_n(x_n^n+1)\frac{\lambda_N^2\omega^2x^2}{2N^2}.\end{aligned} \tag{I.52}$$

After expanding around the Laplacian term around $x_n^n + 1$ and making use of the previous arguments for estimating the remainder and the potential term, one arrive at

$$\widetilde{H}_{n,n+1}(N)\phi_n(x_n^n+1) \geq \phi_n(x_n^n+1)(\lambda'_N\omega(n+\frac{1}{2}) + O(\frac{\lambda_N'^{3/2}}{N})). \tag{I.53}$$

By symmetry, the lower bound in the intervals $x_1^n + 1 < x < \theta N/\sqrt{\lambda'_N}$, $\theta N/\sqrt{\lambda'_N} \leq x < \lfloor N^\Theta \rfloor$ and $\lfloor N^\Theta \rfloor \leq x$ carry on exactly the same as those we did for ϕ_0. This finally concludes the proof: we have obtained that

$$\widetilde{E}_n^k \geq \widetilde{E}(N) \geq \lambda'_N\omega(n+\frac{1}{2}) + O(\frac{\lambda_N'^{3/2}}{N}), \quad k \in \{0,\dots,n\}. \tag{I.54}$$

□

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