Large deviations in the semi-classical limit of quantum spin systems

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Abstract

We prove a full large deviation principle for Gibbs measures arising in the semiclassical limit of quantum spin Hamiltonians and provide an explicit characterization of the associated rate function. In this limit, the probability measures are shown to concentrate on the Riemann sphere.

Introduction

Large deviation theory provides a rigorous framework for analyzing the probabilities of rare events, capturing how these probabilities decay exponentially in the limit of a small parameter [18]. It plays a central role in understanding the concentration of measure phenomena and quantifying the asymptotic behavior of sequences of probability measures. At the heart of this theory lies the *rate function*, a non-negative, lower semicontinuous functional that governs the exponential rate of decay. This function often admits an interpretation analogous to entropy, reflecting the system's fluctuations around its typical behavior.

In statistical mechanics, large deviation principles (LDPs) have been extensively developed for classical lattice spin systems, offering a powerful tool for describing probabilistic behavior in the *thermodynamic limit* [2, 4]. This approach typically analyzes systems as the number of lattice sites or the system volume increases. In contrast, the study of large deviations in the thermodynamic limit of quantum spin systems remains relatively underexplored: with only a few results available, as outlined, for instance, in [5, 8, 12, 13] are available.

Beyond the thermodynamic limit lies another crucial asymptotic regime: the semi-classical limit, which describes the transition from quantum to classical systems. This transition is mathematically captured by quantization theory, offering a rigorous framework in which quantum states are approximated by probability measures on a classical phase space, with the approximation governed by a suitable semiclassical parameter. A natural and compelling question arises: can the rate at which a quantum system approaches its classical counterpart be described within the large deviation framework?

This work directly addresses this question. We focus on the equilibrium states of quantum spin systems, described by the Gibbs state associated with a quantum spin Hamiltonian. These states are parametrized by the *spin quantum number* $N \in \mathbb{N} = \{1, 2, 3, \dots\}$, which serves as an effective semiclassical parameter. Our objective is to establish an LDP for the ensuing probability measures and to characterize the rate of convergence as $N \to \infty$. By doing so, we aim to bridge the gap between quantum statistical mechanics and large deviation theory, offering new insights into the semiclassical behavior of quantum spin systems.

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Spin coherent states

We denote the algebra of bounded operators on $\mathbb{C}^{N+1}$ by $B(\mathbb{C}^{N+1}) \simeq M_{N+1}(\mathbb{C})$, and the associated operator norm by $\|\cdot\|_{\mathrm{op},N}$. We consider a single quantum spin of fixed total angular-momentum $N := 2J \in \mathbb{N}$ and denote by $\boldsymbol{S} = (S_1, S_2, S_3)$ the usual angular momentum operators acting on the $(N+1)$-dimensional complex Hilbert space $\mathbb{C}^{N+1}$, that is

$$[S_1, S_2] = iS_3, \quad \text{and cyclically,}$$

$$S_{\pm} = S_1 \pm iS_2.$$

and

$$S^2 = S_1^2 + S_2^2 + S_3^2 = \frac{N}{2}\left(\frac{N}{2} + 1\right)I,$$

where I always denotes the unit of $M_{N+1}(\mathbb{C})$ for any N. In other words, $\frac{N}{2}\left(\frac{N}{2} + 1\right)$ is the eigenvalue corresponding to the operator S^2.

Analogously, as observed by E. Majorana [9], we may view a single-particle quantum spin system of spin $J = N/2$ as a collective spin system of N spin-$\frac{1}{2}$ particles, subject to permutation invariance. Equivalently, the Hilbert space $\mathbb{C}^{N+1}$ carrying the spin-$J = N/2$ representation is isomorphic to the symmetric subspace

$$\mathcal{H}_N^s := \mathrm{Sym}^N(\mathbb{C}^2) \subset \bigotimes^N \mathbb{C}^2 =: \mathcal{H}_N,$$

and the angular momentum operators (S_1, S_2, S_3) correspond to restrictions of the canonical generators $\boldsymbol{T} = (T_1, T_2, T_3)$ of the N-fold tensor product representation of the Lie-algebra $\mathfrak{su}(2)$, i.e.

$$T_\mu = \sum_{k=1}^N T_\mu^{(k)}, \qquad \mu = 1, 2, 3, \tag{1}$$

with $T_\mu^{(k)}$ acting as the spin Pauli matrix $t_\mu := \frac{1}{2}\sigma_\mu$ on the k-th tensor factor and as the identity on all others. These therefore define bounded operators on $\mathcal{H}_N$, i.e. $T_\mu \in B(\mathcal{H}_N)$. In particular, by permutation invariance

$$S_\mu \cong T_\mu|_{\mathcal{H}_N^s}, \qquad \mu = 1, 2, 3,$$

where $\cong$ should be understood in the sense of unitary equivalence. Note also that T_μ leaves $\mathcal{H}_N^s$ invariant.

On the classical side, we consider the one-point compactification of the complex plane, commonly known as the Riemann sphere $\mathbb{S}^2$. This space can be identified with the complex projective line which naturally carries the structure of a compact Kähler manifold, and can therefore be interpreted as the "physical phase space". This allows the notion of coherent spin states, introduced in what follows.

Let $\Omega = (\theta, \phi)$, has polar angles $\theta \in (0, \pi)$, $\phi \in [0, 2\pi)$. Let $|\uparrow\rangle, |\downarrow\rangle$ be the normalized eigenvectors of σ_3 in $\mathbb{C}^2$, so that

$$\sigma_3|\uparrow\rangle = |\uparrow\rangle, \qquad \sigma_3|\downarrow\rangle = -|\downarrow\rangle,$$

and consider the unit Bloch vector

$$|\Omega\rangle_1 = \cos\!\left(\tfrac{\theta}{2}\right)|\uparrow\rangle + e^{i\phi}\sin\!\left(\tfrac{\theta}{2}\right)|\downarrow\rangle \in \mathbb{C}^2. \tag{2}$$

For $N \in \mathbb{N}$, we define **N-coherent spin vector** $|\Omega\rangle_N \in \mathcal{H}_N^s$, equipped with the usual scalar product inherited from $\mathcal{H}_N$, as

$$|\Omega\rangle_N = \underbrace{|\Omega\rangle_1 \otimes \cdots \otimes |\Omega\rangle_1}_{N \text{ times}}. \tag{3}$$

It is not difficult to see that the map $\mathbb{S}^2 \ni \Omega \mapsto |\Omega\rangle_1 \in \mathbb{C}^2$ lacks continuity at the north pole $\Omega_0 := (0, \phi)$ and south pole $\Omega_\pi = (\pi, \phi)$, where $\phi \in [0, 2\pi)$ is arbitrary. Nonetheless, continuity is preserved at the level of scalar products. To demonstrate this, for $0 \le M \le N$ we introduce the **Dicke state**

$$|M\rangle = |M \downarrow, (N-M) \uparrow\rangle$$

which corresponds to the normalized symmetric vector with M spins in $|\downarrow\rangle$ and $N-M$ spins in $|\uparrow\rangle$ [14, Chapter 7]. The following lemma establishes this property.

LEMMA 1: *Let $A \in B(\mathcal{H}_N^s)$. Then the map*

$$\mathbb{S}^2 \longrightarrow \mathbb{R}, \qquad \Omega \longmapsto \langle\Omega|A|\Omega\rangle_N$$

is continuous. In particular, for $\Omega_0 := (0,\phi)$, $\Omega_\pi := (\pi,\phi)$ and $\phi \in [0,2\pi)$ arbitrary

$$\langle\Omega_0|A|\Omega_0\rangle_N \equiv \lim_{\Omega\to\Omega_0} \langle\Omega|A|\Omega\rangle_N = \langle 0|A|0\rangle_N;$$
$$\langle\Omega_\pi|A|\Omega_\pi\rangle_N \equiv \lim_{\Omega\to\Omega_\pi} \langle\Omega|A|\Omega\rangle_N = \langle N|A|N\rangle_N,$$

Proof. The spin-N coherent (Bloch) states can be expressed in the Dicke basis as

$$|\Omega\rangle_N = \sum_{M=0}^{N} \binom{N}{M}^{1/2} \left(\cos\frac{\theta}{2}\right)^{N-M} \left(\sin\frac{\theta}{2}\right)^{M} e^{\,iM\phi}\,|M\rangle, \tag{4}$$

where $\Omega = (\theta,\phi) \in \mathbb{S}^2$. By direct inspection, for any bounded operator $A \in B(\mathcal{H}_N^s)$, the expectation value $\langle\Omega|A|\Omega\rangle_N$ is continuous on $\mathbb{S}^2\setminus\{(0,\phi),(\pi,\phi)\}$. In the limit $\Omega \to \Omega_0 := (0,\phi)$ (i.e. $\theta \to 0$), we have $\cos(\theta/2) \to 1$, $\sin(\theta/2) \to 0$. Therefore, in the expansion (4), all terms with $M \ge 1$ vanish, and the only surviving contribution is that of $M = 0$. Hence,

$$\lim_{\Omega\to\Omega_0} \langle\Omega|A|\Omega\rangle_N = \langle 0|A|0\rangle_N,$$

which corresponds to the fully polarized Dicke state with all spins up. Similarly, in the limit $\Omega \to \Omega_\pi$, we have $\cos(\theta/2) \to 0$, $\sin(\theta/2) \to 1$. Therefore, in the expansion (4), all terms with $M \le N-1$ vanish, and the only surviving contribution is that of $M = N$. Hence,

$$\lim_{\Omega\to\Omega_\pi} \langle\Omega|A|\Omega\rangle_N = \langle N|A|N\rangle_N,$$

which corresponds to the fully polarized Dicke state with all spins down. $\square$

Berezin maps

For a bounded function $f : \mathbb{S}^2 \to \mathbb{C}$, we denote the usual supremum norm by $\|f\|_\infty$. When $f \in C(\mathbb{S}^2)$ is continuous, we define the **Berezin maps** by

$$Q_N : C(\mathbb{S}^2) \to B(\mathcal{H}_N^s), \qquad f \longmapsto \frac{N+1}{4\pi} \int_{\mathbb{S}^2} d\Omega f(\Omega) |\Omega\rangle\langle\Omega|_N,$$

where $d\Omega$ the unique $SO(3)$-invariant Haar measure on the unit sphere $\mathbb{S}^2$, normalized such that $\int_{\mathbb{S}^2} d\Omega = 4\pi$. The operator $|\Omega\rangle\langle\Omega|_N$ is the one dimensional projection onto the subspace spanned by the N-coherent spin vectors, *cf.* (3). This structure moreover yields, with I being the identity operator on $\mathcal{H}_N^s$ [11]:

$$\begin{aligned} I &= \frac{N+1}{4\pi} \int_{\mathbb{S}^2} d\Omega |\Omega\rangle\langle\Omega|_N, \\ f(\Omega) &= \lim_{N\to\infty} \frac{N+1}{4\pi} \int_{\mathbb{S}^2} d\Omega' f(\Omega') |\langle\Omega|\Omega'\rangle_N|^2. \end{aligned}$$

Quantum spin Hamiltonians

As mentioned above, a single particle quantum spin system may be interpreted as a symmetric tensor product of matrices, restricted to the invariant subspace $\mathcal{H}_N^s$. To clarify this interpretation, we now present a concise overview.

Let $t_1 = \sigma_1/2$, $t_2 = \sigma_2/2$, $t_3 = \sigma_3/2$ be the spin-$\frac{1}{2}$ operators. For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}_0^3$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = L$, consider the L-fold tensor

$$\underbrace{t_1 \otimes \cdots \otimes t_1}_{\alpha_1} \otimes \underbrace{t_2 \otimes \cdots \otimes t_2}_{\alpha_2} \otimes \underbrace{t_3 \otimes \cdots \otimes t_3}_{\alpha_3}.$$

Its fully symmetric version is obtained by applying the normalized symmetrization operator $\mathcal{S}_L$:

$$t_\alpha := \pi_L^L\Big(\underbrace{t_1 \otimes \cdots \otimes t_1}_{\alpha_1} \otimes \underbrace{t_2 \otimes \cdots \otimes t_2}_{\alpha_2} \otimes \underbrace{t_3 \otimes \cdots \otimes t_3}_{\alpha_3}\Big),$$

where π_L^L is defined by Eq. (18) in the Appendix. Each symmetric tensor t_α can be uniquely identified with the monomial [7, Lemma 3.2]

$$p_\alpha(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}.$$

The corresponding symmetric sequence in the N-spin algebra is then defined, for $N \geq L$, as

$$a_N^{(p_\alpha)} := \pi_N^L(t_\alpha), \tag{5}$$

where π_N^L is the embedding map defined in Eq. (18). A general polynomial of degree K,

$$h(x_1, x_2, x_3) = \sum_{L=0}^{K} \sum_{|\alpha|=L} c_\alpha \, x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

with coefficients $c_\alpha \in \mathbb{C}$, corresponds to the symmetric sequence

$$a_N^{(h)} := \sum_{L=0}^{K} \sum_{|\alpha|=L} c_\alpha \, a_N^{(p_\alpha)}.$$

Hence, by defining

$$\bar{H}_N := a_N^{(h)}$$

we generate a *scaled mean-field quantum spin Hamiltonian* $\bar{H}_N$. This operator canonically acts on $\mathcal{H}_N$, but leaves the symmetric subspace $\mathcal{H}_N^s$ invariant. In particular, by restricting $\bar{H}_N$ to $\mathcal{H}_N^s$, we get an operator defined on an $(N+1)$-dimensional Hilbert space. The single particle **quantum spin Hamiltonian** is then defined by

$$H_N^s := N \bar{H}_N|_{\mathcal{H}_N^s}, \tag{6}$$

so that $\|H_N^s\|_{N,op} = O(N)$. From a physical perspective, this implies that the spectral radius scales with the dimension of the underlying space.

Remark 2. To see that the symmetrizer indeed produces a physically meaningful model, we first consider an example. Let $\boldsymbol{T} = (T_1, T_2, T_3)$ be the operators defined in (1). Then

$$T_i = N \, \pi_N^1(t_i),$$

so that, when restricted to the symmetric subspace,

$$T_i|_{\mathcal{H}_N^s} = N \, \pi_N^1(t_i)|_{\mathcal{H}_N^s}.$$

More generally, for any polynomial h in three real variables, the operator

$$h\Big(\frac{2}{N}\boldsymbol{T}\Big)$$

seen as polynomial in the three non-commutative averages satisfies

$$\Big\| h\Big(\frac{2}{N}\boldsymbol{T}\Big)|_{\mathcal{H}_N^s} - \bar{H}_N^s \Big\|_{op,N} = O\Big(\frac{1}{N}\Big), \quad N \to \infty,$$

where $\bar{H}_N^s = H_N^s / N$ is defined above. This explicitly confirms several observations done in [6]. Moreover, as a consequence of [11, Thm. 2.3] and [17], one has the following identification with Berezin quantization, i.e.

$$\|\bar{H}_N^s - Q_N(h|_{\mathbb{S}^2})\|_{op,N} = O\Big(\frac{1}{N}\Big), \quad N \to \infty,$$

where $h|_{\mathbb{S}^2}$ denotes the restriction of the polynomial h to the unit sphere $\mathbb{S}^2$. Hence, the operators $h(\frac{2}{N}\boldsymbol{T})|_{\mathcal{H}_N^s}$ and $Q_N(h|_{\mathbb{S}^2})$ can be seen as (scaled) quantum spin Hamiltonians representing $\bar{H}_N^s$. ■

Classical limit

Let (H_N^s) be a sequence of quantum spin Hamiltonians in $B(\mathcal{H}_N^s)$ that satisfy (6). The local Gibbs state ω_N^β at inverse temperature $\beta > 0$ is defined by

$$\omega_N^\beta(A) := \frac{Tr[e^{-\beta H_N^s} A]}{Tr[e^{-\beta H_N^s}]}, \qquad A \in B(\mathcal{H}_N^s).$$

By direct inspection, one sees that the Berezin map Q_N is *strictly positive* (see also [11, 15]), so that the local Gibbs state ω_N^β at inverse temperature $\beta > 0$ induces a positive linear normalized functional (i.e., a state) on $\mathbb{S}^2$, via

$$\varrho_N^\beta := \omega_N^\beta \circ Q_N.$$

As a matter of fact, plugging in the definition of ω_N^β, this now reads

$$\varrho_N^\beta(f) := \frac{Tr[e^{-\beta H_N^s} Q_N(f)]}{Tr[e^{-\beta H_N^s}]}.$$

By definition of the Berezin map Q_N, the state ϱ_N^β in turn corresponds with a probability measure μ_N^β on $\mathbb{S}^2$, which assumes the following form

$$\mu_N^\beta(U) = \frac{N+1}{4\pi} \int_U d\Omega \frac{\langle \Omega | e^{-\beta H_N^s} | \Omega \rangle}{Tr[e^{-\beta H_N^s}]},$$

where $U \subseteq \mathbb{S}^2$ is a measurable Borel set.

As is known [16, Prop. 4.11], for $f \in C(\mathbb{S}^2)$ arbitrary, the free energy (or pressure) for the function h given above is

$$F_{h,f}^\beta : \mathbb{R} \longrightarrow \mathbb{R}, \qquad t \longmapsto \lim_{N\to\infty} \frac{1}{N} \log \mathrm{Tr}\left(e^{-\beta(H_N^s + tNQ_N(f))} \right)$$

exists for all $t \in \mathbb{R}$. In particular, $F_{h,f}^\beta$ is given by

$$F_{h,f}^\beta(t) = -\inf_{\hat{\mathbb{C}}} (\beta h + tf).$$

On account of [16, Cor. 4.8], the existence of the *classical limit*

$$\varrho^\beta(f) := \lim_{N\to\infty} \varrho_N^\beta(f)$$

is guaranteed provided in addition that $F_{h,f}^\beta$ is differentiable at $t = 0$. In that case, the classical limit is given by

$$\varrho^\beta(f) = \frac{d}{dt}\bigg|_{t=0} F_{h,f}^\beta(t).$$

Indeed, differentiability is guaranteed only in very specific cases [3, 6, 10, 15, 17]. In other words, the classical limit generally does not exist, and this is not an assertion we make.

Nonetheless, one can still attempt to characterize the asymptotic behavior of the sequence of measures μ_N^β. The physical intuition to keep in mind is that the measures concentrate on a suitable subset of the phase space $\mathbb{S}^2$ and outside these sets, they decay exponentially. A central and intriguing question is to determine the precise rate at which this decay occurs. This rate encapsulates the nature of quantum fluctuations and appears to be amenable to rigorous computation. The appropriate mathematical framework for capturing such behavior is provided by the theory of *large deviations*, which will be introduced in the following section.

Principle of large deviations

Below the definition of a large deviation principle is presented.

Definition 3 (LDP)**.** A sequence of probability measures $(\mu_N)_{N\in\mathbb{N}}$ on a Polish space X equipped with the Borel Σ-algebra satisfies a **large deviation principle** (LDP) with a *good rate function* $I : X \to [0, \infty]$, if

(i) I has compact sub-level sets $\{x \in X \mid I(x) \leq k\}$ for all $k \in [0, \infty)$,

(ii) for all compact $C \subseteq X$,

$$\limsup_{N\to\infty} \frac{1}{N} \log \mu_N(C) \leq -\inf_C I,$$

(iii) for all open $O \subseteq X$,

$$\liminf_{N\to\infty} \frac{1}{N} \log \mu_N(O) \geq -\inf_O I.$$

Main result

Our main result is the following. Recall the measures μ_N^β defined for a sequence of self-adjoint Hamiltonians (H_N^s) and $\beta > 0$, defined for measurable subsets U of $X = \mathbb{S}^2$ as

$$\mu_N^\beta(U) = \frac{N+1}{4\pi} \int_U d\Omega \frac{\langle \Omega | e^{-\beta H_N^s} | \Omega \rangle_N}{Tr[e^{-\beta H_N^s}]}.$$

THEOREM 4: *Let a sequence of Hamiltonians $(H_N(s))_N$ assuming the form (6) for h a polynmomial in Ω be given. Then, the sequence of probability measures (μ_N^β) satisfies a large deviation principle with a good rate function $I^\beta : \mathbb{S}^2 \longrightarrow [0, \infty)$*

$$I^\beta(\Omega) := -G^\beta(\Omega) - \beta \inf_{\mathbb{S}^2} h,$$

where G^β is given by the limit

$$G^\beta(\Omega) := \lim_{N\to\infty} \frac{1}{N} \log \langle \Omega | e^{-\beta H_N^s} | \Omega \rangle_N$$

for all $\Omega \in \mathbb{S}^2$.

Before proving the theorem we shortly explain the strategy of the proof to aid the reader. Our strategy is to write the probability measure μ_N^β as

$$d\mu_N^\beta(\Omega) = \frac{N+1}{4\pi} \frac{e^{NG_N^\beta(\Omega)}}{Tr[e^{-\beta H_N^s}]},$$

where

$$G_N^\beta(\Omega) := \frac{1}{N} \log \langle \Omega | e^{-\beta H_N(s)} | \Omega \rangle_N.$$

We will then prove that approximate NG_N^β is almost sub-additive. To this avail, we first take p_α to be a homogeneous monomial of degree L_0, i.e. $p_\alpha(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}$, where $|\alpha| = L_0$. As seen above, this monomial corresponds to a symmetrized tensor of the form

$$t_\alpha := \pi_{L_0}^{L_0}\Big(\underbrace{t_1 \otimes \cdots \otimes t_1}_{\alpha_1} \otimes \underbrace{t_2 \otimes \cdots \otimes t_2}_{\alpha_2} \otimes \underbrace{t_3 \otimes \cdots \otimes t_3}_{\alpha_3}\Big),$$

and the associated symmetric sequence is

$$a_N^{(p_\alpha)} = \pi_N^{L_0}(t_\alpha).$$

Let us abbreviate

$$a_N^s \equiv a_N^{p_\alpha}|_{\mathcal{H}_N^s} = \pi_N^{L_0}(t_\alpha)|_{\mathcal{H}_N^s}.$$

We can now phrase our result in the following general form. For $N, M \geq L_0$ consider the π-sequences (monomials)

$$\begin{aligned}
&a_N := \pi_N^{L_0}(t_\alpha); \quad a_N^s := \pi_N^{L_0}(t_\alpha)|_{\mathcal{H}_N^s};\\
&a_M := \pi_M^{L_0}(t_\alpha); \quad a_M^s := \pi_M^{L_0}(t_\alpha)|_{\mathcal{H}_M^s};\\
&a_{N+M} := \pi_{N+M}^{L_0}(t_\alpha); \quad a_{N+M}^s := \pi_{N+M}^{L_0}(t_\alpha)|_{\mathcal{H}_{N+M}^s}.
\end{aligned}$$

This leads to the following lemma.

Lemma 5: *With the notation introduced above, for all $N, M \geq L_0$, it holds*

$$\langle \Omega, e^{(N+M)a_{N+M}^s} \Omega \rangle_{N+M} \leq C \langle \Omega, e^{Na_N^s} \Omega \rangle_N \langle \Omega, e^{Ma_M^s} \Omega \rangle_M, \tag{7}$$

for some constant $C = C(L_0, N, M)$ with $C \geq 1$ and $C = O(1)$, as $N, M \to \infty$. In particular,

$$\lim_{N\to\infty} \frac{1}{N} \log \langle \Omega, e^{Na_N^s} \Omega \rangle_N = \inf_N \frac{1}{N} \log \langle \Omega, e^{Na_N^s} \Omega \rangle_N. \tag{8}$$

Proof. We start with the following observation. Restricting the symmetric sequence (a_N) to $\mathcal{H}_N^s$ corresponds to the operator $a_N p_N$, where p_N is an orthogonal projection onto $\mathcal{H}_N^s$. Since the coherent state $|\Omega\rangle_N \in \mathcal{H}_N^s$, and a_N leaves $\mathcal{H}_N^s = \operatorname{ran}(p_N)$ invariant, it follows that

$$\langle \Omega, e^{Na_N}\Omega\rangle_N = \langle \Omega, e^{Na_N^s}\Omega\rangle_N,$$

i.e., it does not matter whether we restrict the symmetric sequence a_N to $\mathcal{H}_N^s$ or not. This allows us to work directly with the sequence (a_N), and prove (7) for a_N.

Now we consider right-hand side of (7). We first observe, for any matrices $a_N \in B^N$ and $b_M \in B^M$, it holds

$$e^{a_N} \otimes e^{b_M} = (e^{a_N} \otimes 1_M)(1_N \otimes e^{Nb_M}) = e^{a_N \otimes 1_M} e^{1_N \otimes b_M} = e^{a_N \otimes 1_M + 1_N \otimes b_M}.$$

If we apply the product state $|\Omega\rangle_{N+M}$, one finds

$$\langle \Omega, e^{a_N}\Omega\rangle_N \langle \Omega, e^{b_M}\Omega\rangle_M = \langle \Omega, e^{a_N \otimes 1_M + 1_N \otimes b_M}\Omega\rangle_{N+M}$$

We use the following identity: for all $N, M \geq L_0$

$$\begin{aligned} a_{N+M} &= \pi_{N+M}^{L_0}(t_\alpha) = \pi_{N+M}(\pi_N^{L_0}(t_\alpha) \otimes 1_M) = \pi_{N+M}(a_N \otimes 1_M) \\ &= \pi_{N+M}(a_N \otimes 1_M) = \pi_{N+M}(1_N \otimes \pi_M^{L_0}(t_\alpha)). \end{aligned}$$

This implies that for symmetric sequences $(a_N)_N$,

$$\pi_{N+M}(a_N \otimes 1_M + 1_N \otimes a_M) = 2a_{N+M},$$

In particular,

$$\pi_{N+M}(Na_N \otimes 1_M + 1_N \otimes Ma_M) = (N+M)a_{N+M}.$$

Therefore, the first assertion (7) holds true whenever there exists a bounded constant $C = C(N, M, L_0)$ with $C \geq 1$ and $C = O(1)$, such that

$$\begin{aligned} &\langle \Omega^{N+M}, e^{\pi_{N+M}(Na_N \otimes 1_M + 1_N \otimes Ma_M)}\Omega^{N+M}\rangle \\ &\leq C\langle \Omega^{N+M}, e^{Na_N \otimes 1_M + 1_N \otimes Ma_M}\Omega^{N+M}\rangle, \end{aligned} \tag{9}$$

whenever $N, M \geq L_0$. To prove (9), let us denote by

$$X := Na_N \otimes 1_M + 1_N \otimes Ma_M.$$

In this notation, (9) reads

$$\langle \Omega, e^{\pi_{N+M}(X)}\Omega\rangle_{N+M} \leq C\langle \Omega, e^X\Omega\rangle_{N+M}.$$

We first prove the following claim.
Claim:
The symmetrization over the full permutation group $G := S_{N+M}$ satisfies

$$\pi_{N+M}(X) = \alpha_X(L_0, N, M)\, X + R_X,$$

where $\alpha_X(L_0, N, M)$ is a coefficient and R_X denotes the *interaction terms*, i.e., the components of the symmetrized operator that do not preserve the tensor block structure of X. More precisely, there exists an L_0, N, M-dependent constant $E(L_0, N, M)$, *cf.* (11) below, such that

- R_X has norm bounded by

  $$\|R_X\| \leq E(L_0, N, M)\,(N + M)$$

- The coefficients $\alpha_X(L_0, N, M)$ satisfy

  $$\alpha_X(L_0, N, M) = 1 - E(L_0, N, M) \to 1$$

  uniformly as $N, M \to \infty$.

Proof:
We can group the total number of permutations in G as follows:

(I) permutations that only act on the N-block, not acting on the M-block;

(II) permutations that do not act on the N-block, acting only on the M-block;

(III) permutations that only permute identities between the N and M block, keeping the non-trivial tensors untouched;

(IV) permutations that only interchange non-trivial tensors between the N and M blocks, leaving the identities untouched.

In this way, $\alpha_X(L_0, N, M)$ corresponds to the number of permutations in G that leave the operator X invariant, i.e. to group (I),(II) and (III). The interaction terms (group IV) arise from permutations that map at least one nontrivial operator a_i originally supported in the N-block into the M-block, or vice versa, while at least one other nontrivial operator remains in the N-block. By assumption, since L_0 is the total number of nontrivial tensor factors in X, the number of permutations corresponding to such interactions is

$$\sum_{k=1}^{L_0-1} \binom{M}{k} \binom{L_0}{k} k!,$$

where

- $\binom{M}{k}$ counts the ways to assign k nontrivial operators to the M-block,
- $\binom{L_0}{k}$ counts the ways to pick k nontrivial operators out of the L_0.

The extreme cases $k = 0$ and $k = L_0$ are excluded since they correspond to all nontrivial operators residing entirely within one block, thus preserving the form of X and not contributing to interactions. Similarly, the contribution from permutations originally assigning nontrivial operators in the M-block and mapping some into the N-block is given by

$$\sum_{k=1}^{L_0-1} \binom{N}{k}\binom{L_0}{k} k!.$$

Their total is bounded above by

$$\sum_{k=1}^{L_0-1} \binom{M}{k}\binom{L_0}{k} k! + \sum_{k=1}^{L_0-1} \binom{N}{k}\binom{L_0}{k} k! \leq C(L_0)\binom{N+M}{L_0}, \tag{10}$$

where the last inequality follows from the Vandermonde identity and $C(L_0)$ only depends on L_0, not on M and N. Hence, the total fraction of the interaction terms in the symmetrization is given by

$$E(L_0, N, M) := \frac{1}{|G|}\sum_{k=1}^{L_0-1} \binom{M}{k}\binom{L_0}{k} k! + \frac{1}{|G|}\sum_{k=1}^{L_0-1} \binom{N}{k}\binom{L_0}{k} k!. \tag{11}$$

$$\leq \frac{C(L_0)}{L_0!(N+M-L_0)!}, \tag{12}$$

As a result, we indeed have the following decomposition

$$\pi_{N+M}(X) = \alpha_X(L_0, N, M)X + R_X.$$

Note that indeed,

- X has norm bounded by $\|X\| \leq N + M$ due to its prefactors N and M and the fact that the t_α are bounded by one;
- R_X collects the interaction terms of norm bounded by $\|R_X\| \leq E(L_0, N, M)(N+M)$;
- $\alpha(L_0, N, M) = 1 - E(L_0, N, M) \to 1$, uniformly as $N, M \to \infty$.

This proves the claim. ■

To continue our prove we use Duhamel's integral formula, stating that for any square matrices A and B

$$e^{A+B} - e^A = \int_0^1 e^{(1-s)A} B e^{s(A+B)} ds.$$

In particular, for the choices $A = X$, $B = R_X$, and $\alpha = \alpha_X(L_0, N, M)$, we set $C = (\alpha - 1)A + B$ and obtain the following estimate

$$\|e^{\pi_{N+M}(X)} - e^X\| = \|e^{\alpha A + B} - e^X\| \leq \int_0^1 \|e^{(1-s)A} C e^{s(A+C)}\| ds \leq$$
$$\int_0^1 \|e^{(1-s)A} C e^{s(A+C)}\| ds \leq \|C\| e^{\|A\| + \|C\|}.$$

By the previous claim, $e^{\|A\|+\|C\|} = O(e^{N+M})$, $\|B\| = O(\frac{N+M}{(N+M-L_0)!})$ and $1 - \alpha = E(L_0, N, M) = O(\frac{1}{(N+M-L_0)!})$. It follows that

$$\begin{aligned}
\gamma(L_0, N, M) &:= \|C\| e^{\|A\|+\|C\|} \\
&\leq (E(L_0, N, M)\|A\| + \|B\|) e^{\|A\|+\|C\|} \\
&= O\Big(\frac{N+M}{(N+M-L_0)!}\Big) O(e^{N+M}) \to 0, \quad (N, M \to \infty).
\end{aligned}$$

By Cauchy Schwarz,

$$\Big|\langle \Omega, e^{\pi_{N+M}(X)} \Omega \rangle_{N+M} - \langle \Omega, e^X \Omega \rangle_{N+M}\Big| \leq \|e^{\pi_{N+M}(X)} - e^X\| \leq \gamma(L_0, N, M),$$

so that

$$\begin{aligned}
&\langle \Omega, e^{\pi_{N+M}(X)} \Omega \rangle_{N+M} \leq \langle \Omega, e^X \Omega \rangle_{N+M} + \gamma(L_0, N, M) \\
&= \langle \Omega, e^X \Omega \rangle_{N+M} \Big(1 + \frac{\gamma(L_0, N, M)}{\langle \Omega, e^X \Omega \rangle_{N+M}}\Big)
\end{aligned}$$

for all $N, M \geq L_0$. If we set

$$C(L_0, N, M) := 1 + \frac{2\gamma(N, M, L_0)}{\langle \Omega, e^X \Omega \rangle_{N+M}} \tag{13}$$

then

$$C(L_0, N, M) \to 1,$$

as $N, M \to \infty$ uniformly, since the denominator decays at worst exponentially, whilst the numerator decays super-exponentially (see above). If we set

$$C := \sup_{N,M \geq L_0} C(L_0, N, M),$$

then $1 \leq C < \infty$ uniformly in N, M. This concludes the proof of equation (9).

To prove (8) we rely on Fekete's theorem. To this avail, let us abbreviate

$$z_N := \log \langle \Omega, e^{N a_N} \Omega \rangle_N.$$

From (9), the sequence $(z_N)_{N\in\mathbb{N}}$ satisfies

$$z_{M+N} \leq \log C + z_M + z_N,$$

for some bounded constant with $\log C \geq 0$ and all $M, N \geq L_0$. Moreover, by Jensen's inequality,

$$\frac{z_N}{N} \geq \langle \Omega, a_N \Omega \rangle_N.$$

The limit of the right-hand side exists as $N \to \infty$: it corresponds to the polynomial $p_K(\Omega)$, see e.g. [15]. As a result, the sequence $(\frac{z_N}{N})_N$ is bounded from below. We now claim that the limit

$$\lim_{N\to\infty} \frac{z_N}{N} = \inf_{N\in\mathbb{N}} \frac{z_N}{N}$$

actually exists. To see this, define the sequence $v_N := \log C + z_N$. Then

$$v_{M+N} = \log C + z_{M+N} \leq \log C + (\log C + z_M + z_N) = v_M + v_N.$$

Thus, $(v_N)_N$ is subadditive. We may apply Fekete's lemma, and conclude that the limit

$$L := \lim_{N\to\infty} \frac{v_N}{N} = \inf_{N\in\mathbb{N}} \frac{v_N}{N}$$

exists. This implies that

$$\lim_{N\to\infty} \frac{z_N}{N} = \lim_{N\to\infty} \left(\frac{v_N}{N} - \frac{\log C}{N} \right) = L.$$

Clearly, $|L| < \infty$, since $(\frac{z_N}{N})$ is bounded from below by a converging sequence. This proves the lemma. $\square$

The following result extends the previous lemma to generic polynomials in three real variables.

COROLLARY 6: *Let h be polynomial of degree K,*

$$h(x_1, x_2, x_3) = \sum_{L=0}^{K} \sum_{|\alpha|=L} c_\alpha \, x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

with coefficients $c_\alpha \in \mathbb{C}$, and consider the ensuing mean-field quantum spin Hamiltonian H_N^s, as defined by (6). Then, for all $N, M \geq K$ it holds

$$\langle \Omega, e^{H_{N+M}^s} \Omega \rangle_{N+M} \leq C \langle \Omega, e^{H_N^s} \Omega \rangle_N \langle \Omega, e^{H_M^s} \Omega \rangle_M,$$

for some bounded constant $C \geq 1$ depending on K, but not on N and M. In particular,

$$\lim_{N\to\infty} \frac{1}{N} \log \langle \Omega, e^{H_N^s} \Omega \rangle_N = \inf_N \frac{1}{N} \log \langle \Omega, e^{H_N^s} \Omega \rangle_N. \tag{14}$$

Since the proof goes in an exact similar way as the one in Lemma 5, it is omitted.

Proof of Theorem 4. For the lower bound we proceed as follows. for any measurable (open) set $U \subset \mathbb{S}^2$ it holds

$$\frac{1}{N}\log \mu_N^\beta(U) = \frac{1}{N}\log\frac{N+1}{\pi} + \frac{1}{N}\log\int_U d\Omega e^{NG_N^\beta(\Omega)} - \frac{1}{N}\log Tr[e^{-\beta H_N(s)}]$$

Taking the liminf yields

$$\liminf_{N\to\infty}\frac{1}{N}\log\mu_N^\beta(U) = \liminf_{N\to\infty}\frac{1}{N}\log\int_U d\Omega e^{NG_N^\beta(\Omega)} + \beta\inf_{\mathbb{S}^2} h$$

On account of Lemma 5, for all $N \in \mathbb{N}$ and $\Omega \in \mathbb{S}^2$ it holds $G_N^\beta(\Omega) \geq G^\beta(\Omega)$, and so is $e^{NG_N^\beta(\Omega)} \geq e^{NG^\beta(\Omega)}$ for any Ω. Integrating both sides over U with respect to the (normalized) measure $d\Omega$ gives

$$\frac{1}{N}\log\int_U d\Omega e^{NG_N^\beta(z)} \geq \frac{1}{N}\log\int_U d\Omega e^{NG^\beta(\Omega)}.$$

We set

$$I^\beta(\Omega) := -G^\beta(\Omega) - \beta\inf_{\mathbb{S}^2} h,$$

which is lower-semi continuous, since $G^\beta(\Omega)$ is upper semi continuous. By Laplace Principle, the right-hand side satisfies

$$\lim_{N\to\infty}\frac{1}{N}\log\int_U d\Omega e^{NG^\beta(\Omega)} = \sup_U(G^\beta) = -\inf_U(-G^\beta),$$

Therefore, we obtain

$$\liminf_{N\to\infty}\frac{1}{N}\log\mu_N^\beta(U) \geq -\inf_U I^\beta$$

implying the lower bound of the claim.

To prove the upper bound we proceed as follows. Fix $\varepsilon > 0$ and $C \subset \mathbb{S}^2$ compact. Since $G^\beta(\Omega) = \inf_N G_N^\beta(\Omega)$, for each $\Omega \in C$ there exists $N_\Omega \in \mathbb{N}$ such that

$$G_{N_\Omega}^\beta(\Omega) \leq G^\beta(\Omega) + \varepsilon.$$

Because each G_N^β is continuous on $\mathbb{S}^2$, for every $\Omega \in C$ we can find an open neighborhood V_Ω of Ω such that

$$\sup_{\Omega'\in V_\Omega} G_{N_\Omega}^\beta(\Omega') \leq G_{N_\Omega}^\beta(\Omega) + \varepsilon \leq G^\beta(\Omega) + 2\varepsilon \leq \sup_{\Omega\in C} G^\beta(\Omega) + 2\varepsilon$$

The family $\{V_\Omega : \Omega \in C\}$ covers C. By compactness, choose a finite subcover $V_{\Omega_1}, \dots, V_{\Omega_m}$ with associated integers $N_{\Omega_1}, \dots, N_{\Omega_m}$. For each $i = 1, \dots, m$ we then have

$$\sup_{\Omega \in V_{\Omega_i}} G^\beta_{N_{\Omega_i}}(\Omega) \le \sup_{\Omega \in C} G^\beta(\Omega) + 2\varepsilon.$$

Let N_0 denote a common multiple of $N_{\Omega_1}, \dots, N_{\Omega_m}$, e.g. their least common multiple. For each i there exists $k_i \ge 1$ such that $N_0 = k_i N_{\Omega_i}$. Let us furthermore write $\tilde{G}^\beta_{N_0}(\Omega) := G^\beta_{N_0}(\Omega)$ Using sub-additivity repeatedly (as in the proof of the Lemma 5 with constant $C_0 > 0$) we obtain, for every $\Omega \in \mathbb{S}^2$ and each i,

$$\tilde{G}^\beta_{N_0}(\Omega) \le k_i \tilde{G}^\beta_{N_{\Omega_i}}(\Omega) + k_i \log C_0,$$

and hence, dividing by N_0,

$$G^\beta_{N_0}(\Omega) \le G^\beta_{N_{\Omega_i}}(\Omega) + \max_{1 \le i \le m} \frac{\log C_0}{N_{\Omega_i}}$$

Because the sets $V_{\Omega_1}, \dots, V_{\Omega_m}$ cover C, for every $\Omega \in C$ there exists i such that $\Omega \in V_{\Omega_i}$. For such Ω we have

$$G^\beta_{N_0}(\Omega) \le \sup_{\Omega' \in V_{\Omega_i}} G^\beta_{N_{\Omega_i}}(\Omega') + \frac{\log C_0}{N_{\Omega_i}}$$

Using the previous estimate $\sup_{V_{\Omega_i}} G^\beta_{N_{\Omega_i}} \le \sup_C G^\beta + 2\varepsilon$, we obtain

$$G^\beta_{N_0}(\Omega) \le \sup_C G^\beta + 2\varepsilon + \max_{1 \le i \le m} \frac{\log C_0}{N_{\Omega_i}}.$$

Now fix $N \ge N_0$ and write $N = kN_0 + r$ with $0 \le r \le N_0 - 1$. Sub-additivity gives, for all $\Omega \in \mathbb{S}^2$,

$$\tilde{G}^\beta_N(\Omega) \le \tilde{G}^\beta_{kN_0}(\Omega) + \tilde{G}^\beta_r(\Omega) + \log C_0,$$

and again,

$$\tilde{G}^\beta_{kN_0}(\Omega) \le k\tilde{G}^\beta_{N_0}(\Omega) + (k-1)\log C_0.$$

Combining both and exponentiating yields, for every $\Omega \in C$,

$$e^{NG^\beta_N(\Omega)} = e^{\tilde{G}^\beta_N(\Omega)} \le e^{k\tilde{G}^\beta_{N_0}(\Omega)} \, e^{\tilde{G}^\beta_r(\Omega)} \, e^{k \log C_0}.$$

Using the bound $G^\beta_{N_0}(\Omega) \le \sup_C G^\beta + 2\varepsilon + \max_{1 \le i \le m} \frac{\log C_0}{N_{\Omega_i}}$ valid on C, we get

$$e^{NG^\beta_N(\Omega)} \le e^{kN_0(\sup_C G^\beta + 2\varepsilon + \max_{1 \le i \le m} \frac{\log C_0}{N_{\Omega_i}})} \, e^{\tilde{G}^\beta_r(\Omega)} \, e^{k \log C_0}.$$

Integrating over C, taking logarithms, and dividing by $N = kN_0 + r$ gives

$$\frac{1}{N}\log\int_C e^{NG_N^\beta(\Omega)}\,d\Omega \le \frac{1}{kN_0+r}\log\int_C e^{\tilde{G}_r^\beta(\Omega)}\,d\Omega + \sup_C G^\beta + 2\varepsilon + \max_{1\le i\le m}\frac{\log C}{N_{\Omega_i}} + \frac{\log C}{N_0} \tag{15}$$

The first term on the right-hand side of (15) is bounded by

$$\frac{1}{kN_0+r}\big(\log\mathrm{Vol}(C) + \|\tilde{G}_r^\beta\|_\infty\big),$$

which vanishes as $k\to\infty$ because $r \le N_0 - 1$ is fixed. Hence letting $N\to\infty$ (equivalently $k\to\infty$ with fixed N_0) we find

$$\limsup_{N\to\infty}\frac{1}{N}\log\int_C e^{NG_N^\beta(\Omega)}\,d\Omega \le \sup_C G^\beta + 2\varepsilon + \max_{1\le i\le m}\frac{\log C}{N_{\Omega_i}} + \frac{\log C}{N_0}$$

Choose the N_{Ω_i} sufficiently large such that $\max_{1\le i\le m}\frac{\log C}{N_{\Omega_i}} < \varepsilon$. Then, since $N_0 \ge N_{\Omega_i}$, also $\frac{\log C}{N_0} < \varepsilon$. Thus

$$\limsup_{N\to\infty}\frac{1}{N}\log\int_C e^{NG_N^\beta(\Omega)}\,d\Omega \le \sup_C G^\beta + 3\varepsilon.$$

Recalling the definition of the probability measure

$$\mu_N^\beta(C) = \frac{1}{Tr[e^{-\beta H_N^s}]}\int_C e^{NG_N^\beta(\Omega)}\,d\Omega,$$

we obtain

$$\limsup_{N\to\infty}\frac{1}{N}\log\mu_N^\beta(C) \le \sup_C G^\beta + \beta\inf_{\mathbb{S}^2} h + 3\varepsilon.$$

Defining the rate function $I^\beta(\Omega) := -G^\beta(\Omega) - \beta\inf_{\mathbb{S}^2} h$, we have

$$\limsup_{N\to\infty}\frac{1}{N}\log\mu_N^\beta(C) \le -\inf_C I^\beta + 3\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the claim follows:

$$\limsup_{N\to\infty}\frac{1}{N}\log\mu_N^\beta(C) \le -\inf_C I^\beta.$$

We now prove the requested features of I^β. Since $G_N^\beta : \mathbb{S}^2 \to \mathbb{R}$ is continuous (Lemma 1), the pointwise limit G^β is upper semi continuous, and hence $I^\beta = -G^\beta - \inf_{\mathbb{S}^2} h$ is lower semi continuous. To show that $I^\beta \ge 0$, we prove that

$$G^\beta \le -\beta\inf_{\mathbb{S}^2} h.$$

Indeed, on account of [15, Thm. 6.1.2] the smallest eigenvalue λ_N of $\bar{H}_N(s)$ converges to $\inf_{\mathbb{S}^2} h$. Hence, for each $\epsilon > 0$, we can find N large enough such that

$$G_N^\beta(\Omega) \leq ||G_N^\beta||_{N,op} = \frac{1}{N}\log e^{-\beta N\lambda_N} = -\beta\lambda_N = -\beta \inf_{\mathbb{S}^2} h + \epsilon.$$

In particular,

$$G^\beta(\Omega) \leq -\beta \inf_{\mathbb{S}^2} h.$$

Furthermore I^β is a good rate function, since $\mathbb{S}^2$ is compact and sub-level sets are closed.

We finally show that I^β vanishes at the points minimizing the symbol h.

$$I^\beta(\Omega) = -G^\beta(\Omega) - \beta \inf_{\mathbb{S}^2} h.$$

Furthermore, by the continuity of the bundle, Jensen's inequality, and the inequality $G^\beta \leq -\beta \inf h$, the choice of such a z minimizing h leads to

$$\begin{aligned} &-\beta \inf_{\mathbb{S}^2} h = -\beta h(\Omega) = \lim_{N\to\infty} -\frac{\beta}{N}\langle\Omega, H_N(s)\Omega\rangle_N \\ &\leq \lim_{N\to\infty}\frac{1}{N}\log\langle\Omega, e^{-\beta H_N(s)}\Omega\rangle_N = \lim_{N\to\infty} G_N^\beta(\Omega) = G^\beta(\Omega) \leq -\beta\inf_{\mathbb{S}^2} h, \end{aligned} \tag{16}$$

□

Remark 7. We point out that the set $\{I^\beta = 0\} = \{z \mid I^\beta(z) = 0\}$ contains the minimum set of h. Indeed, on account of the proof of Theorem 4, particularly the final chain of inequalities, we know that for $\Omega \in \{h = \inf_{\mathbb{S}^2}\} = \{z \mid h(\Omega) = \inf_{\mathbb{S}^2} h\}$, it holds $I^\beta(\Omega) = 0$, and hence

$$\{h = \inf_{\mathbb{S}^2} h\} \subseteq \{I^\beta = 0\}.$$

This implies that, even if the sequence of probability measures (μ_N^β) does not converge, it still concentrates around the minimizers of h. ■

A Symmetric sequences

Let B be a unital C^*-algebra, e.g. the matrix algebra $M_2(\mathbb{C})$. The *symmetrization operator* $\pi_N : B^N \to B^N$ is defined as the unique linear continuous extension of the following map on elementary tensors:

$$\pi_N(a_1 \otimes \cdots \otimes a_N) := \frac{1}{N!}\sum_{\tau\in\mathcal{P}(N)} a_{\tau(1)} \otimes \cdots \otimes a_{\tau(N)}. \tag{17}$$

Furthermore, for $N \geq M$ we need to generalize the definition of S_N to give a bounded operator $\pi_N^M : B^M \to B^N$, defined by linear and continuous extension of

$$\pi_N^M(b) := \pi_N(b \otimes \underbrace{I \otimes \cdots \otimes I}_{N-M\text{times}}), \quad b \in B^{\otimes M}. \tag{18}$$

A sequence $(a_N)_N$ is called **symmetric** if there exist $M \in \mathbb{N}$ and $a_M \in B^{\otimes M}$ such that

$$a_N = \pi_N^M(a_M) \text{ for all } N \geq M, \tag{19}$$

and **quasi-symmetric** if $a_{1/N} = \pi_N(a_{1/N})$ if $N \in \mathbb{N}$, and for every $\epsilon > 0$, there is a symmetric sequence $(b_{1/N})_{N\in\mathbb{N}}$ as well as $M \in \mathbb{N}$ (both depending on ϵ) such that

$$\|a_{1/N} - b_{1/N}\| < \epsilon \text{ for all } N > M. \tag{20}$$

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