Global observables in statistical mechanics

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Abstract

This note presents a canonical construction of global observables - sometimes referred to in the literature as macroscopic observables or observables at infinity- in statistical mechanics, providing a unified treatment of both commutative and non-commutative cases. Unlike standard approaches, the framework is formulated directly in the C^*-algebraic setting, without relying on any specific representation.

1 Introduction

In the statistical mechanics of lattice systems, one typically studies large collections of interacting particles arranged on a regular lattice, and the behavior of these systems in appropriate limiting regimes reveals their macroscopic properties. Two distinct types of such "infinite volume" limits are commonly considered.

The first is the *thermodynamic limit*, in which the lattice size tends to infinity while local observables are examined within finite regions of the lattice. This limit is described by the so-called quasi-local algebra and allows one to rigorously define expectation values of local quantities and ensures that bulk properties, such as energy density or correlation functions, stabilize in the infinite-volume system. Its significance is well established in the context of phase transitions and equilibrium phenomena in quantum and classical statistical mechanics [3, 4, 11].

The second, less standard but equally important, is what is sometime referred to as *macroscopic limit* [2, 13], which focuses on observables that are often called "global" or "observables at infinity," such as spatial averages of local quantities over regions of diverging size [9, 10, 12]. Unlike the thermodynamic limit, which addresses the stability of local observables under volume growth, the macroscopic limit captures collective, non-local behavior that reflects emergent classical features of quantum systems [7, 14].

While this construction is classical in spirit and well established in statistical mechanics, its quantum counterpart is considerably less developed, often requiring sophisticated representation-theoretic methods [3, 4].

In this work we develop a C^*-algebraic framework to study the macroscopic behavior of statistical mechanical systems, providing a unified treatment of both commutative and non-commutative settings. This approach emphasizes the connection between quantum and classical systems and does not require the use of any particular representation. In the non-commutative setting, the resulting C^*-algebra naturally contains commutative C^*- subalgebras generated by macroscopic averages, thereby providing a bridge between classical thermodynamics as a limit of quantum statistical mechanics. In the commutative case, it is proved that the ensuing C^*-algebra can be identified with the well-known algebra of *σ-tail measurable functions*, underscoring the significance of this construction.

C^*-product and quotient algebra

Let $\Gamma \subset \mathbb{R}^d$ be a countable set. This already endows the set of finite subsets of Γ with a partial order (inclusion), which is upward directed and hence defines a notion of convergence, namely "convergence along the net of finite subsets of Γ, directed by inclusion", that is

$$\lim_{\Lambda \nearrow \Gamma} F(\Lambda) = \alpha$$

means that for every $\varepsilon > 0$, there exists a finite subset $K_\epsilon \subset \Gamma$ such that

$$\|F(\Lambda) - \alpha\| < \varepsilon \quad \text{whenever } K_\epsilon \subset \Lambda.$$

Here, F is a function (or operator) defined on finite subsets of Γ and taking values in a normed space.[1]

To each $x \in \Gamma$ we associate a unital C^*-algebra $\mathcal{B}_x$; in fact, we assume the same algebra $\mathcal{B}$ for all x, and use x only to denote the lattice position. The (minimal) tensor product of $\otimes_{x\in\Lambda}\mathcal{B}_x$ is denoted by $\mathcal{B}_\Lambda$ and ensures that $\mathcal{B}_\Lambda$ is again a C^*-algebra. We then consider the **full C^*-product** $\prod_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda$ defined by

$$\prod_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda := \{(a_\Lambda)_\Lambda \,|\, (\|a_\Lambda\|_\Lambda)_\Lambda \in \ell^\infty(\Gamma)\}\,, \tag{1}$$

where $(a_\Lambda)_\Lambda$ should be understood as element in the algebraic direct product with pointwise operations. As it is well-known [1] $\prod_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda$ is a C^*-algebra

[1]This notion of convergence should not be confused with convergence in the sense of van Hove.

with respect to sup norm $\|(a_\Lambda)_\Lambda\|_\infty := \sup_{\Lambda\Subset\Gamma}\|a_\Lambda\|_\Lambda$. For $(a_\Lambda)_\Lambda, (b_\Lambda)_\Lambda \in \Pi_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda$, we introduce the following **$\sim$-equivalence relation**

$$(a_\Lambda)_\Lambda \sim (b_\Lambda)_\Lambda \Longleftrightarrow \lim_{\Lambda\nearrow\Gamma}\|a_\Lambda - b_\Lambda\|_\Lambda = 0\,. \tag{2}$$

For each given sequence $(a_\Lambda)_\Lambda$, we will denote by $[a_\Lambda]_\Lambda := [(a_\Lambda)_\Lambda]$ the corresponding equivalence class with respect to (2). Moreover, the **direct C^*-sum**

$$\bigoplus_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda := \{(a_\Lambda)_\Lambda \in \prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda \,|\, (\|a_\Lambda\|_\Lambda)_\Lambda \in C_0(\Gamma)\} \tag{3}$$

$$= \{(a_\Lambda)_\Lambda \in \prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda \,|\, \lim_{\Lambda\nearrow\Gamma}\|a_\Lambda\|_\Lambda = 0\} \tag{4}$$

is a closed two-sided ideal in $\prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda$ and thus we may consider the quotient

$$[\mathcal{B}]_\sim := \prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda / \bigoplus_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda\,, \tag{5}$$

which is nothing but the space of $\sim$-equivalence classes $[a_\Lambda]_\Lambda$ for bounded sequences $(a_\Lambda)_\Lambda$, i.e.

$$[\mathcal{B}]_\sim = q(\prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda)$$

where q is the canonical quotient map,

$$q : \prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda \longrightarrow \prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda / \bigoplus_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda, \quad q((a_\Lambda)_{\Lambda\Subset\Gamma}) = [(a_\Lambda)_{\Lambda\Subset\Gamma}]_\Lambda,$$

Importantly, $[\mathcal{B}]_\sim$ is a C^*-algebra with norm

$$\|[a_\Lambda]_\Lambda\|_{[\mathcal{B}]_\sim} = \limsup_{\Lambda\nearrow\Gamma}\|a_\Lambda\|_\Lambda\,. \tag{6}$$

Hence, passing to the quotient makes it possible to identify sequences that represent the same observable in the limit as $\Lambda \nearrow \Gamma$, so that only essentially distinct observables are captured. Of course, $[\mathcal{B}]_\sim$ is very large, so that one typically focuses on suitable C^*-sub algebras.

Example 1 (Quasi-local algebra): We consider the $*$-algebra of local observables

$$\dot{\mathcal{B}}^\infty := \bigcup_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda.$$

Here, we implicitly (injectively) embedded $\mathcal{B}_\Lambda$ into the full C^*- product

$$\prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda$$

by identifying an element $b_{\Lambda'} \in \mathcal{B}_{\Lambda'}$ with a local sequence $(b_\Lambda)_{\Lambda \Subset \Gamma}$ defined as

$$b_\Lambda := \begin{cases} b_{\Lambda'} \otimes \mathbf{1}_{\Lambda \setminus \Lambda'}, & \Lambda' \subset \Lambda, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathbf{1}_{\Lambda\setminus\Lambda'}$ denotes the identity in $\mathcal{B}_{\Lambda\setminus\Lambda'} = \otimes_{x\in\Lambda\setminus\Lambda'}\mathcal{B}_x$.

The algebra $\dot{\mathcal{B}}^\infty$ is a sub-algebra of $\prod_{\Lambda\Subset\Gamma}\mathcal{B}_\Lambda$. The **quasi-local algebra** is then defined to be the completion of the quotient

$$[\mathcal{B}]^\infty := \overline{q(\dot{\mathcal{B}}^\infty)}^{\|\cdot\|} \subset [\mathcal{B}]_\sim,$$

where norm is given by $\|[a_\Lambda]_\Lambda\|_{[\mathcal{B}]^\infty} = \limsup_{\Lambda\nearrow\Gamma}\|a_\Lambda\|_\Lambda$, which can be shown to equal the actual limit. It is not difficult to see that $[\mathcal{B}]^\infty$ is a C^*-subalgebra of $[\mathcal{B}]_\sim$. The quasi-local algebra provides the standard framework for describing the thermodynamic limit in statistical mechanics [3, 4]. ■

The C^*-algebra $[\mathcal{B}]_\sim$ contains many other elements for which the expectation value is not defined in several physically relevant states, such as translation-invariant states. We give some examples of such sequences.

Example 2: *We consider the following sequences of tensor products of matrices:*

(i)

$$a_\Lambda := \bigotimes_{x\in\Lambda}\sigma_x^1,$$

where σ_x^1 is the 1-component of the Pauli matrix at site x.

(ii)

$$a_\Lambda := \bigotimes_{x\in\Lambda}\begin{cases}\sigma_x^1, & x \text{ odd},\\ \sigma_x^3, & x \text{ even}.\end{cases}$$

(iii) Define a sequence $(a_\Lambda)_\Lambda$ by partitioning Γ into consecutive blocks of strictly increasing lengths $(B_n)_{n\geq 0}$, assigning σ^x to even-indexed blocks and σ^z to odd-indexed blocks, and setting

$$a_\Lambda := \bigotimes_{x\in\Lambda} A_x, \quad A_x := \begin{cases}\sigma^x & \text{if } x \text{ is in an even-indexed block},\\ \sigma^z & \text{if } x \text{ is in an odd-indexed block}.\end{cases}$$

For a translation-invariant state ω, the limit

$$\lim_{\Lambda\nearrow\Gamma}\omega(a_\Lambda)$$

is generally not defined for these sequences, because the product over infinitely many fluctuating operators diverges or oscillates.

C^*-algebraic construction of global observables in quantum statistical mechanics

We now construct another C^*-sub-algebra of $[\mathcal{B}]_\sim$. Let

$$\mathcal{C}^\infty := \left\{ (a_\Lambda)_\Lambda \in \prod_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda : \forall \quad (b_\Lambda)_\Lambda \in \dot{\mathcal{B}}^\infty, \ \lim_{\Lambda \nearrow \Gamma} \|[a_\Lambda, b_\Lambda]\|_\Lambda = 0 \right\}.$$

LEMMA 3: *$\mathcal{C}^\infty$ is a norm-closed *-subalgebra of $\prod_\Lambda \mathcal{B}_\Lambda$, and hence a C^*-algebra.*

Proof. We must prove the following three conditions.

(1) $*$-algebra property: Let $a = (a_\Lambda)_\Lambda, a' = (a'_\Lambda)_\Lambda \in \mathcal{C}^\infty$ and $b = (b_\Lambda)_\Lambda \in \dot{\mathcal{B}}^\infty$. Then

$$[a+a', b] = [a, b]+[a', b], \quad [a^*, b] = [a, b^*]^*, \quad [aa', b] = a[a', b]+[a, b]a'.$$

Since $\|[a_\Lambda, b_\Lambda]\| \to 0$ and $\|[a'_\Lambda, b_\Lambda]\| \to 0$, the same holds for $a + a'$, a^*, and aa'. Thus $\mathcal{C}^\infty$ is a *-subalgebra.

(2) Norm-closedness: Let $(a^{(n)}) \subset \mathcal{C}^\infty$ be a net that converges in the sup-norm to a, i.e. $\|a^{(n)} - a\|_\infty \to 0$. Then for any $b \in \dot{\mathcal{B}}^\infty$ and each Λ,

$$\|[a_\Lambda, b_\Lambda]\| \leq \|[a_\Lambda - a^{(n)}_\Lambda, b_\Lambda]\| + \|[a^{(n)}_\Lambda, b_\Lambda]\| \leq 2\|a - a^{(n)}\|_\infty \|b_\Lambda\| + \|[a^{(n)}_\Lambda, b_\Lambda]\|.$$

Taking the limit $\Lambda \nearrow \Gamma$ and then $n \to \infty$, the right-hand side goes to 0. Hence $a \in \mathcal{C}^\infty$.

(3) C^*-property: Being a closed *-subalgebra of the C*-algebra $\prod_\Lambda \mathcal{B}_\Lambda$ with the sup-norm, $\mathcal{C}^\infty$ is itself a C^*-algebra.

$\square$

Consider the quotient

$$[\mathcal{C}]^\infty := \mathcal{C}^\infty / \bigoplus_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda = q(\mathcal{C}^\infty),$$

where q is the canonical projection onto $[\mathcal{B}]_\sim$. It follows that $[\mathcal{C}]^\infty$ is a C^*-algebra, and

$$[\mathcal{C}]^\infty = [\mathcal{B}]_\sim \cap (q(\dot{\mathcal{B}}^\infty))',$$

is the relative commutant of the $*$-algebra $[\dot{\mathcal{B}}]^\infty$ in the C^*-algebra $[\mathcal{B}]_\sim$, and therefore norm-closed. Furthermore, we may define

$$\mathcal{C}^\infty_\Lambda := \left\{ (a_{\Lambda'})_{\Lambda'} \in \prod_{\Lambda' \Subset \Gamma} \mathcal{B}_{\Lambda'} : \forall \quad b_\Lambda \in \mathcal{B}_\Lambda, \ \lim_{\Lambda' \nearrow \Gamma} \|[a_{\Lambda'}, b_\Lambda]\|_{\Lambda'} = 0 \right\},$$

where b_Λ has to be understood as element of $\mathcal{B}_{\Lambda'}$ under the canonical embedding. It then follows

$$\mathcal{C}^\infty = \bigcap_{\Lambda \Subset \Gamma} \mathcal{C}^\infty_\Lambda.$$

Motivated by the pioneering ideas described in [9, Sec. 2.3.6], we introduce the following definition.

DEFINITION 4: *The C^*-subalgebra $[\mathcal{C}]^\infty$ is called the* ***algebra of global observables****.* ■

The C^*-algebra $[\mathcal{C}]^\infty$ turns out to be non-commutative.

LEMMA 5: $[\mathcal{C}]^\infty$ *is non-commutative.*

Proof. Consider the sequences $(a_\Lambda)_\Lambda$ and $(c_\Lambda)_\Lambda$, defined by

$$a_\Lambda := \mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes a_{x_\Lambda}, \quad c_\Lambda := \mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes c_{x_\Lambda},$$

where $a_{x_\Lambda}, c_{x_\Lambda}$ are fixed non-commuting observables acting on site $x_\Lambda \in \Lambda$, i.e., local observables in $\mathcal{B}_{\{x_\Lambda\}}$, such that $[a_{x_\Lambda}, c_{x_\Lambda}] \neq 0$. These sequences are what we call "local sequences translated to infinity," where x_Λ is chosen so that $x_\Lambda \to \infty$ as $\Lambda \nearrow \Gamma$.

For any local observable $b \in \dot{\mathcal{B}}^\infty$, supported on a finite subset $\Lambda' \Subset \Gamma$, define the sequence

$$b_\Lambda := \begin{cases} b_{\Lambda'} \otimes \mathbf{1}_{\Lambda\setminus\Lambda'}, & \Lambda' \subseteq \Lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Since a_Λ acts non-trivially only on site x_Λ which eventually is supported outside Λ' and b_Λ acts non-trivially only on sites in Λ', their supports are disjoint for all sufficiently large Λ. Hence,

$$[a_\Lambda, b_\Lambda] = 0 \quad \text{for all sufficiently large } \Lambda.$$

Therefore,

$$\lim_{\Lambda \nearrow \Gamma} \|[a_\Lambda, b_\Lambda]\| = 0,$$

and similarly for c_Λ. This shows

$$(a_\Lambda)_\Lambda, (c_\Lambda)_\Lambda \in \dot{\mathcal{C}}^\infty.$$

By definition,

$$[a_\Lambda, c_\Lambda] = (\mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes a_{x_\Lambda})(\mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes c_{x_\Lambda}) - (\mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes c_{x_\Lambda})(\mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes a_{x_\Lambda}).$$

Using standard tensor product properties, we find

$$[a_\Lambda, c_\Lambda] = \mathbf{1}_{\Lambda\setminus\{x_\Lambda\}} \otimes [a_{x_\Lambda}, c_{x_\Lambda}].$$

Applying the operator norm, we obtain

$$\|[a_\Lambda, c_\Lambda]\|_\Lambda = \|[a_{x_\Lambda}, c_{x_\Lambda}]\|,$$

where the norm on the right-hand side does not depend on Λ. Recall that the norm on the quotient algebra $[\mathcal{C}]^\infty$ is defined as

$$\|[a, c]\|_{[\mathcal{C}]^\infty} := \limsup_{\Lambda \nearrow \Gamma} \|[a_\Lambda, c_\Lambda]\|.$$

Since $\|[a_\Lambda, c_\Lambda]\|_\Lambda = \|[a_{x_\Lambda}, c_{x_\Lambda}]\|$ for all Λ, and the commutator $[a_{x_\Lambda}, c_{x_\Lambda}] \neq 0$, we conclude that

$$\|[a, c]\|_{[\mathcal{C}]^\infty} > 0.$$

Therefore, the elements $a = [a_\Lambda]_\Lambda$ and $c = [c_\Lambda]_\Lambda$ in $[\mathcal{C}]^\infty$ do *not* commute. As a result, the algebra $[\mathcal{C}]^\infty$ is non-commutative. □

Besides the observables "translated to infinity" considered in Lemma 5, we give another example of a global observable.

EXAMPLE 6: *Set $\Gamma = \mathbb{Z}$. Consider the following sequence of local tensor products:*

$$c_N := \bigotimes_{k=1}^{N-\lfloor N/2 \rfloor} 1 \otimes \bigotimes_{j=1}^{\lfloor N/2 \rfloor} a,$$

where 1 denotes the identity operator on a single site and a is a non-trivial and non-zero single-site matrix with $\|a\| \leq 1$. This sequence looks like

$$\begin{aligned}
c_1 &= 1, \\
c_2 &= 1 \otimes a, \\
c_3 &= 1 \otimes 1 \otimes a, \\
c_4 &= 1 \otimes 1 \otimes a \otimes a, \\
c_5 &= 1 \otimes 1 \otimes 1 \otimes a \otimes a, \\
c_6 &= 1 \otimes 1 \otimes 1 \otimes a \otimes a \otimes a, \\
c_7 &= 1 \otimes 1 \otimes 1 \otimes 1 \otimes a \otimes a \otimes a, \\
c_8 &= 1 \otimes 1 \otimes 1 \otimes 1 \otimes a \otimes a \otimes a \otimes a.
\end{aligned}$$

Note that the sequence $(c_N)_N$ is uniformly bounded, and for any fixed local observable b, c_N asymptotically commutes with b. Therefore, $(c_N)_N$ defines a legitimate element in $[\mathcal{C}]^\infty$. However $(c_N)_N$ does in general not commute with the sequences constructed in the proof Lemma 5. ■

In the next section, we will prove that $[\mathcal{C}]^\infty$ admits an interesting commutative C^*-subalgebra, namely the one generated by "macroscopic averages".

Commutative subalgebras

We construct a C^*-subalgebra of $[\mathcal{C}]^\infty$, containing all "macroscopic averages". This algebra enjoys the property of being commutative, and therefore resembles a classical observable algebra describing the macroscopic limit arising from the underlying quantum statistical mechanics.

To illustrate this idea, we focus on the one-dimensional lattice i.e., $\Gamma = \mathbb{Z}$. Let $\Lambda_1 \subset \Lambda_2 \subset \dots$ be a strictly increasing sequence of connected finite subsets of $\mathbb{Z}$, with $|\Lambda_N| = N$ and $\cup_N \Lambda_N = \mathbb{Z}$. For each region $\Lambda = \Lambda_N$, we consider the linear operator (left-shift operator)

$$\gamma_\Lambda \colon \mathcal{B}_\Lambda \to \mathcal{B}_\Lambda,$$

uniquely defined by continuous and linear extension of the following map on elementary tensors:

$$\gamma_\Lambda(a_1 \otimes \dots \otimes a_N) := a_2 \otimes \dots \otimes a_N \otimes a_1\,, \tag{7}$$

where $a_1, \dots, a_N \in \mathcal{B}$. The operator γ_Λ is a $*$-endomorphism of the algebra $\mathcal{B}_\Lambda$. Moreover, $\gamma_\Lambda^N = \mathrm{id}$. We then define the **averaged shift operator**

$$\overline{\gamma}_\Lambda := \frac{1}{N}\sum_{j=0}^{N-1} \gamma_\Lambda^j\,. \tag{8}$$

The image of this operator,

$$\mathcal{B}_\Lambda^\gamma := \overline{\gamma}_\Lambda(\mathcal{B}_\Lambda), \tag{9}$$

is a C^*-subalgebra of $\mathcal{B}_\Lambda$ consisting of the γ-invariant elements.

Definition 7: *A sequence $(a_\Lambda)_\Lambda$ is called a* ***γ-sequence*** *if there exists a finite subset $\Lambda_0 \Subset \Gamma$ and an element $a_{\Lambda_0} \in \mathcal{B}_{\Lambda_0}$ such that*

$$a_\Lambda = \overline{\gamma}_\Lambda^{\Lambda_0}(a_{\Lambda_0}) := \begin{cases} \overline{\gamma}_\Lambda\left(\mathbf{1}_{\Lambda\setminus\Lambda_0} \otimes a_{\Lambda_0}\right), & \Lambda \supseteq \Lambda_0, \\ 0, & \textit{otherwise}, \end{cases} \tag{10}$$

where $\mathbf{1}_{\Lambda\setminus\Lambda_0}$ denotes the identity on the complementary tensor factors. ■

In what follows, we consider the $*$-algebra $\dot{\mathcal{B}}_\gamma^\infty \subset \prod_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda$ generated by all γ-sequences, together with its image under the canonical projection $[\dot{\mathcal{B}}]_\gamma^\infty \subset [\mathcal{B}]_\sim^\infty$. The algebra $[\dot{\mathcal{B}}]_\gamma^\infty$ enjoys remarkable properties; in particular, it can be completed to a commutative C^*-algebra $[\mathcal{B}]_\gamma^\infty$ [7, Prop. 6]. Moreover, the γ-sequences constitute a continuous field of C^*-algebras over $\mathbb{N} \cup \{\infty\}$ with limit $[\mathcal{B}]_\gamma^\infty$.

We begin with the following observation, which relates states on $[\mathcal{B}]_\gamma^\infty$ to translationally invariant states on the quasi-local algebra.

Remark 8: Each state on $[\mathcal{B}]^\infty_\gamma$ canonically induces a translationally invariant state on the quasi-local algebra $[\mathcal{B}]^\infty$. Indeed, one can prove that the map

$$\begin{aligned}\overline{\gamma}_\infty : [\mathcal{B}]^\infty &\to [\mathcal{B}]^\infty_\gamma;\\ [a_\Lambda]_\Lambda &\mapsto [\overline{\gamma}_\Lambda(a_\Lambda)]_\Lambda\end{aligned}$$

is well-defined. In particular, applying a state $\omega \in S([\mathcal{B}]^\infty_\gamma)$ to an equivalence class of γ-sequences, yields

$$\omega([\overline{\gamma}_\Lambda(a_\Lambda)]) = \omega \circ \overline{\gamma}_\infty([a_\Lambda]_\Lambda)$$

If now we define

$$\hat{\omega} := \omega \circ \overline{\gamma}_\infty,$$

then $\hat{\omega}$ is a translationally invariant state on $[\mathcal{B}]^\infty$, since for any j

$$\hat{\omega} \circ \gamma^j_\infty = \hat{\omega},$$

where

$$\gamma^j_\infty([a_\Lambda]_\Lambda) = [\gamma^j_\Lambda(a_\Lambda)]_\Lambda$$

is the translation of j lattice sites.

In this way, the expectation values of γ-sequences in $\hat{\omega}$ capture the macroscopic averages of local observables and thereby encode the distribution over ergodic components of the corresponding translation-invariant state, see [3, Chapter 4] for further details hereon. ■

More importantly, $[\mathcal{B}]^\infty_\gamma$ is a C^*-subalgebra of $[\mathcal{C}]^\infty$.

Proposition 9: $[\mathcal{B}]^\infty_\gamma \subset [\mathcal{C}]^\infty$.

Proof. The assertion follows directly from the construction. We first prove it for γ-sequences; the general case then follows immediately from [7, Prop. 6]. For any γ-sequence $(a_\Lambda)_\Lambda$ as in Definition 7, we claim that for all local observables $(b_\Lambda)_\Lambda \in \dot{\mathcal{B}}^\infty$

$$\lim_{\Lambda \nearrow \Gamma} \|[a_\Lambda, b_\Lambda]\|_\Lambda = 0. \tag{11}$$

To see this, fix $\Lambda_0 \Subset \Gamma$ and $a_{\Lambda_0} \in \mathcal{B}_{\Lambda_0}$ such that

$$a_\Lambda = \overline{\gamma}^{\Lambda_0}_\Lambda(a_{\Lambda_0}) = \frac{1}{N}\sum_{j=0}^{N-1} \gamma^j_\Lambda(1_{\Lambda\setminus\Lambda_0} \otimes a_{\Lambda_0}), \qquad \Lambda \supseteq \Lambda_0, \tag{12}$$

and $a_\Lambda = 0$ otherwise. Let $(b_\Lambda)_\Lambda$ be a local observable supported on some fixed $\Lambda' \Subset \Gamma$. Since b_Λ acts non-trivially only on the tensor factors associated with Λ', and a_Λ is an average over cyclic shifts, the commutator norm can be estimated as

$$\|[a_\Lambda, b_\Lambda]\|_\Lambda = \left\| \frac{1}{N} \sum_{j=0}^{N-1} \left[\gamma_\Lambda^j (1_{\Lambda \setminus \Lambda_0} \otimes a_{\Lambda_0}), b_\Lambda \right] \right\|_\Lambda \tag{13}$$

$$\leq \frac{1}{N} \sum_{j=0}^{N-1} \left\| \left[\gamma_\Lambda^j (1_{\Lambda \setminus \Lambda_0} \otimes a_{\Lambda_0}), b_\Lambda \right] \right\|_\Lambda . \tag{14}$$

For large Λ, most of the shifts γ_Λ^j move the support of a_{Λ_0} away from the support of b_Λ. More precisely, at most $|\Lambda_0|+|\Lambda'|$ terms in the sum correspond to non-disjoint supports, and hence potentially non-zero commutators. For all other terms, the supports are disjoint, so the commutator vanishes. Thus,

$$\|[a_\Lambda, b_\Lambda]\|_\Lambda \leq \frac{|\Lambda_0| + |\Lambda'|}{N} \, 2\|a_{\Lambda_0}\| \, \|b_\Lambda\|_\Lambda. \tag{15}$$

Since $(b_\Lambda)_\Lambda$ is a fixed local observable, $\|b_\Lambda\|_\Lambda$ is uniformly bounded. Hence,

$$\|[a_\Lambda, b_\Lambda]\|_\Lambda = O(1/N),$$

and thus

$$\lim_{\Lambda \nearrow \Gamma} \|[a_\Lambda, b_\Lambda]\|_\Lambda = 0. \tag{16}$$

This shows that every γ-sequence $(a_\Lambda)_\Lambda$ belongs to $\mathcal{C}^\infty$. Passing to equivalence classes via the canonical quotient map

$$q : \prod_{\Lambda \Subset \Gamma} \mathcal{B}_\Lambda \;\to\; [\mathcal{B}]_\sim, \tag{17}$$

we obtain

$$[a_\Lambda]_\Lambda = q((a_\Lambda)_\Lambda) \in [\mathcal{C}]^\infty. \tag{18}$$

□

Remark 10: The C^*-algebra $[\mathcal{B}]_\gamma^\infty$ properly contains the commutative C^*- subalgebra $[\mathcal{B}]_\pi^\infty$, that is, the algebra of equivalence classes generated by permutation symmetric sequences. In particular, $[\mathcal{B}]_\pi^\infty \cong C(S(\mathcal{B}))$ [7, 14], which is a direct consequence of the quantum De Finetti theorem. ■

C^*-algebraic construction of global observables in classical statistical mechanics

Consider again the discrete set $\Gamma \subset \mathbb{R}^d$. To each $x \in \Gamma$ we associate a unital commutative C^*-algebra endowed with a Poisson bracket, $(\mathcal{A}_x, \{\cdot, \cdot\})$. This structure plays an important role in equilibrium thermodynamics within classical statistical mechanics, as it provides a natural framework for formulating the classical KMS condition [8].

For any finite subset $\Lambda \subset \Gamma$, we denote the tensor product $\bigotimes_{x\in\Lambda} \mathcal{A}_x$ by $\mathcal{A}_\Lambda$, and equip it with the local supremum norm $\|\cdot\|_\Lambda$. To define the Poisson bracket, we first fix a dense Poisson $*$-subalgebra $\dot{\mathcal{A}}_x \subset \mathcal{A}_x$ and then define $\dot{\mathcal{A}}_\Lambda$ accordingly for each finite subset $\Lambda \Subset \Gamma$. Then, in a fashion similar to above, the algebra

$$\dot{\mathcal{D}}^\infty := \left\{ (a_\Lambda)_\Lambda \in \prod_{\Lambda \Subset \Gamma} \dot{\mathcal{A}}_\Lambda : \ \forall (b_\Lambda)_\Lambda \in \dot{\mathcal{A}}^\infty, \ \lim_{\Lambda \nearrow \Gamma} \|\{a_\Lambda, b_\Lambda\}_\Lambda\|_\Lambda = 0 \right\},$$

where $\dot{\mathcal{A}}^\infty$ denotes the Poisson algebra of all local sequences, is a commutative $*$-subalgebra of $\prod_{\Lambda\Subset\Gamma} \mathcal{A}_\Lambda$. For a finite region $\Lambda \Subset \Gamma$, we define

$$\dot{\mathcal{D}}^\infty_\Lambda := \Big\{ (a_{\Lambda'})_{\Lambda'} \in \prod_{\Lambda' \Subset \Gamma} \dot{\mathcal{A}}_{\Lambda'} \ : \ \forall b_\Lambda \in \dot{\mathcal{A}}_\Lambda, \ \lim_{\Lambda' \nearrow \Gamma} \|\{a_{\Lambda'}, b_\Lambda\}_{\Lambda'}\|_{\Lambda'} = 0 \Big\},$$

where b_Λ is embedded canonically into $\mathcal{A}_{\Lambda'}$ for $\Lambda' \supseteq \Lambda$. Analogous to the algebra $\mathcal{C}^\infty$, it follows that

$$\dot{\mathcal{D}}^\infty = \bigcap_{\Lambda \Subset \Gamma} \dot{\mathcal{D}}^\infty_\Lambda.$$

Moreover, we can complete $\dot{\mathcal{D}}^\infty$ with respect to the norm $\|\cdot\| := \sup_{\Lambda\Subset\Gamma} \|\cdot\|_\Lambda$, yielding the C^*-algebra

$$\mathcal{D}^\infty := \overline{\dot{\mathcal{D}}^\infty}^{\|\cdot\|}.$$

The pertinent quotient

$$[\mathcal{D}]^\infty := \mathcal{D}^\infty / \bigoplus_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda$$

with $\bigoplus_{\Lambda\Subset\Gamma} \mathcal{A}_\Lambda$ the ideal of vanishing sequences, is the classical analog of $[\mathcal{C}]^\infty$, and forms a C^*-subalgebra of the quotient C^*-algebra $[\mathcal{A}]_\sim$. Note that this algebra can be also obtained from $\dot{\mathcal{D}}^\infty$, as follows from

$$\mathcal{D}^\infty / \bigoplus_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda = \overline{\dot{\mathcal{D}}^\infty / \bigoplus_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda}^{\|\cdot\|},$$

with norm given by (6).

Remark 11: We recall the tail-σ-algebra from classical statistical mechanics:

$$\mathscr{T}_\infty := \bigcap_{\Lambda \Subset \Gamma} \mathscr{F}_{\Lambda^c}, \tag{19}$$

where $\mathscr{F}_{\Lambda^c}$ is the σ-algebra of events that only depend on “spins” located outside Λ. The algebra $[\mathcal{C}]^\infty$ resembles its quantum analog, canonically formalized in a C^*-algebraic way, i.e. without passing to any representation. ■

We have the following result, connecting $[\mathcal{D}]^\infty$ with tail-measurable functions.

PROPOSITION 12: *Let X be a connected finite dimensional symplectic manifold and consider the C^*-algebra $\mathcal{A} = C_0(X)$. Let Γ be a countable index set and suppose $\mathcal{A}_x = C_0(X)$ for all $x \in \Gamma$. Define the set*

$$\Omega := \prod_{x \in \Gamma} X_x,$$

equipped with the product topology. Then, for any equivalence class $[a] \in [\mathcal{D}]^\infty$, there exists a unique bounded function

$$f_a : \Omega \to \mathbb{C}$$

which is measurable with respect to the tail σ-algebra

$$\mathscr{T}_\infty := \bigcap_{\Lambda \Subset \Gamma} \sigma(\{\omega_x : x \in \Gamma \setminus \Lambda\}).$$

Conversely, every function $f : \Omega \to \mathbb{C}$ that is bounded and measurable with respect to the tail σ-algebra induces a net $(a_\Delta)_\Delta$ of bounded functions $a_\Delta \in \ell^\infty(\Omega_\Delta)$, with $(\|a_\Delta\|)_{\Delta \Subset \Gamma} \in \ell^\infty(\Gamma)$, such that, as $\Delta \nearrow \Gamma$, the support associated with any local observable eventually becomes disjoint from the support of a_Δ. The net $(a_\Delta)_\Delta$ is unique after fixing a reference configuration.

Proof. Note that $\dot{\mathcal{D}}^\infty / \bigoplus_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda$ is dense in $[\mathcal{D}]^\infty$. Given an equivalence class $[a] \in [\mathcal{D}]^\infty$ and $\epsilon > 0$, find $[\dot{a}] \in \dot{\mathcal{D}}^\infty / \bigoplus_{\Lambda \Subset \Gamma} \mathcal{A}_\Lambda$, such that

$$\|[a] - [\dot{a}]\| = \limsup_{\Lambda' \nearrow \Gamma} \|a_{\Lambda'} - \dot{a}_{\Lambda'}\|_{\Lambda'} < \epsilon.$$

Based on this, we prove the result for $[\dot{a}]$ taken to be self-adjoint. The latter is sufficient, since each element decomposes into a sum of two self-adjoint elements. To this end, let us fix a (self-adjoint) representative net $(\dot{a}_{\Lambda'})_{\Lambda'} \in \dot{\mathcal{D}}^\infty$. We can view it as a real-valued function on

$$\Omega = \prod_{x \in \Gamma} X_x$$

via the pull-back of the canonical projection $\omega \mapsto \pi_{\Lambda'}(\omega) = \omega|_{\Lambda'} \equiv \omega_{\Lambda'}$, i.e.,

$$\dot{a}_{\Lambda'}(\omega) := \dot{a}_{\Lambda'}(\omega_{\Lambda'}) \quad \text{for all } \omega \in \Omega.$$

We then define a function $f_{\dot{a}} : \Omega \to \mathbb{R}$ by

$$f_{\dot{a}}(\omega) := \limsup_{\Lambda' \nearrow \Gamma} \dot{a}_{\Lambda'}(\omega).$$

This function is well defined and hence unique for the given equivalence class: if $(\dot{b}_{\Lambda'})_{\Lambda'}$ is another representative, then

$$\lim_{\Lambda' \nearrow \Gamma} \|\dot{b}_{\Lambda'} - \dot{a}_{\Lambda'}\|_{\Lambda'} = 0,$$

and hence, for all ω,

$$|f_{\dot{a}}(\omega) - f_{\dot{b}}(\omega)| \leq \limsup_{\Lambda' \nearrow \Gamma} \|\dot{a}_{\Lambda'} - \dot{b}_{\Lambda'}\|_{\Lambda'} = \lim_{\Lambda' \nearrow \Gamma} \|\dot{a}_{\Lambda'} - \dot{b}_{\Lambda'}\|_{\Lambda'} = 0.$$

Since $\sup_{\Lambda'} \|\dot{a}_{\Lambda'}\|_{\Lambda'} < \infty$ (the sequence belongs to ℓ^∞), we have

$$\|f_{\dot{a}}\|_\infty \leq \sup_{\Lambda'} \|\dot{a}_{\Lambda'}\|_{\Lambda'} < \infty.$$

We now prove that $f_{\dot{a}}$ is tail-measurable. Fix any $\Lambda_0 \Subset \Gamma$. Note first that if g is any smooth compactly supported local self-adjoint element with $\mathrm{supp}(g) \subset \Lambda_0$, its Hamiltonian vector field X_g is complete, and the (globally defined for all t) flow φ_t^g only acts on the local coordinates $\omega_{\Lambda_0} \in \Omega_{\Lambda_0}$. For $\Lambda' \supseteq \Lambda_0$, let $g_{\Lambda'}$ denote the canonical embedding of g into $\dot{\mathcal{A}}_{\Lambda'}$. Likewise, each $\dot{a}_{\Lambda'} \in \dot{\mathcal{A}}_{\Lambda'}$ is a smooth local observable depending only on $\omega_{\Lambda'}$. For any $\omega \in \Omega$, with $\omega_{\Lambda'} = \omega|_{\Lambda'}$, the chain rule then gives

$$\frac{d}{dt}(\dot{a}_{\Lambda'}(\varphi_t^{g_{\Lambda'}}(\omega_{\Lambda'}))) = \{\dot{a}_{\Lambda'}, g_{\Lambda'}\}(\varphi_t^{g_{\Lambda'}}(\omega_{\Lambda'})).$$

Integrating in t we obtain

$$\dot{a}_{\Lambda'}(\varphi_t^{g_{\Lambda'}}(\omega_{\Lambda'})) - \dot{a}_{\Lambda'}(\omega_{\Lambda'}) = \int_0^t \{\dot{a}_{\Lambda'}, g_{\Lambda'}\}(\varphi_s^{g_{\Lambda'}}(\omega_{\Lambda'}))\, ds.$$

This implies the uniform estimate

$$\sup_{\omega_{\Lambda'}} \left| \dot{a}_{\Lambda'}(\varphi_t^{g_{\Lambda'}}(\omega_{\Lambda'})) - \dot{a}_{\Lambda'}(\omega_{\Lambda'}) \right| \leq |t|\, \|\{\dot{a}_{\Lambda'}, g_{\Lambda'}\}\|_{\Lambda'}.$$

Taking $\limsup_{\Lambda' \nearrow \Gamma}$ and using $(\dot{a}_{\Lambda'})_{\Lambda'} \in \dot{\mathcal{D}}^\infty$, the right-hand side tends to 0. It follows that for any t in a compact set and any $\omega_{\Lambda'} \in \Omega_{\Lambda'}$

$$\limsup_{\Lambda' \nearrow \Gamma} \left| \dot{a}_{\Lambda'}(\varphi_t^{g_{\Lambda'}}(\omega_{\Lambda'})) - \dot{a}_{\Lambda'}(\omega_{\Lambda'}) \right| = 0.$$

By the definition of $f_{\dot{a}}$ this entails

$$f_{\dot{a}}(\Phi_t^g(\omega)) = f_{\dot{a}}(\omega), \quad \text{for all } t \in \mathbb{R},$$

where $\Phi_t^g(\omega) := (\varphi_t^g(\omega_{\Lambda_0}), \omega_{\Gamma \setminus \Lambda_0})$. This expression is well defined, since g is localized on Ω_{Λ_0}, and hence its flow acts only on coordinates inside Λ_0. Equivalently, for any configuration $\omega \in \Omega$,

$$f_{\dot{a}}(\varphi_t^g(\omega_{\Lambda_0}), \omega_{\Gamma \setminus \Lambda_0}) = f_{\dot{a}}(\omega_{\Lambda_0}, \omega_{\Gamma \setminus \Lambda_0}), \quad \text{for all } t \in \mathbb{R},$$

This $f_{\dot{a}}$ is invariant under local Hamiltonian flows supported in Λ_0. To conclude, we take $\omega, \rho \in \Omega$ with $\omega|_{\Gamma\setminus\Lambda_0} = \rho|_{\Gamma\setminus\Lambda_0}$. By standard constructions, the group generated by compactly supported Hamiltonian diffeomorphisms acts transitively on a connected, symplectic manifold. Hence, any two local configurations can be connected by a finite composition of local Hamiltonian flows supported in Λ_0. Hence, there exists a finite sequence of smooth local Hamiltonians $g_1, \ldots, g_n$ supported in Λ_0 and times $t_1, \ldots, t_n$ such that

$$\rho_{\Lambda_0} = \varphi_{t_n}^{g_n} \circ \cdots \circ \varphi_{t_1}^{g_1}(\omega_{\Lambda_0}).$$

Since $f_{\dot{a}}$ is invariant under each $\varphi_{t_j}^{g_j}$, we obtain

$$f_{\dot{a}}(\rho_{\Lambda_0}, \rho_{\Gamma\setminus\Lambda_0}) = f_{\dot{a}}(\omega_{\Lambda_0}, \omega_{\Gamma\setminus\Lambda_0}),$$

and hence $f_{\dot{a}}$ does not depend on coordinates inside any local region. Thus,

$$\forall \Lambda_0 \Subset \Gamma,\ \forall \omega, \rho \in \Omega,\ \omega|_{\Gamma\setminus\Lambda_0} = \rho|_{\Gamma\setminus\Lambda_0} \implies f_{\dot{a}}(\omega) = f_{\dot{a}}(\rho).$$

This is exactly the definition of tail-measurability. Finally, note that since

$$\|f_a - f_{\dot{a}}\|_\infty \le \limsup_{\Lambda' \nearrow \Gamma} \|a_{\Lambda'} - \dot{a}_{\Lambda'}\|_{\Lambda'} < \epsilon,$$

the same result holds for f_a, i.e. the function induced by the equivalence class $[a]$. This proves the forward implication.

For the converse, let f be $\mathscr{T}_\infty$–measurable. By definition, f is measurable with respect to every $\mathscr{F}_{\Gamma\setminus\Lambda}$. This means that for every finite subset $\Lambda \Subset \Gamma$ there exists a $\mathscr{F}_{\Gamma\setminus\Lambda}$–measurable function

$$f_\Lambda : \prod_{x\in\Gamma\setminus\Lambda} X_x \to \mathbb{C}$$

such that for all $\omega \in \Omega$,

$$f(\omega) = f_\Lambda(\omega|_{\Gamma\setminus\Lambda}).$$

These functions are automatically consistent: if $\Lambda \subseteq \Lambda'$, then

$$f_\Lambda = f_{\Lambda'} \circ r_{\Lambda',\Lambda}, \tag{20}$$

where $r_{\Lambda',\Lambda} : \Omega_{\Gamma\setminus\Lambda} \to \Omega_{\Gamma\setminus\Lambda'}$ is the natural restriction map. Indeed, since $f = f_\Lambda \circ \pi_{\Gamma\setminus\Lambda} = f_{\Lambda'} \circ \pi_{\Gamma\setminus\Lambda'}$ and $\pi_{\Gamma\setminus\Lambda'} = r_{\Lambda',\Lambda} \circ \pi_{\Gamma\setminus\Lambda}$, equality (20) follows from the surjectivity of the projection maps $\pi_{\Gamma\setminus\Lambda}$.

Fix $\Delta \Subset \Gamma$. Choose any finite subset $\Lambda_\Delta \Subset \Gamma$. From the above, we have $f = f_{\Lambda_\Delta} \circ \pi_{\Gamma\setminus\Lambda_\Delta}$. By the consistency relation

$$f_{\Lambda_\Delta} = f_{\Lambda'} \circ r_{\Lambda',\,\Lambda_\Delta} \qquad \text{for all } \Lambda' \supset \Lambda_\Delta.$$

It follows that for all $\tilde{\omega}, \tilde{\omega}' \in \Omega$ satisfying

$$\tilde{\omega}|_{\Gamma\setminus\Lambda_\Delta} = \tilde{\omega}'|_{\Gamma\setminus\Lambda_\Delta},$$

we have

$$f(\tilde{\omega}) = f_{\Lambda_\Delta}(\pi_{\Gamma\setminus\Lambda_\Delta}(\tilde{\omega})) = f_{\Lambda_\Delta}(\pi_{\Gamma\setminus\Lambda_\Delta}(\tilde{\omega}')) = f(\tilde{\omega}')$$

Set

$$K_\Delta := \Delta \cap (\Gamma \setminus \Lambda_\Delta) = \Delta \setminus \Lambda_\Delta.$$

Fix a reference configuration $\xi \in \Omega$ and define, for $\omega_\Delta \in \Omega_\Delta$, a configuration $\eta \in \Omega_{\Gamma\setminus\Lambda_\Delta}$ by

$$\eta|_{K_\Delta} = \omega_\Delta|_{K_\Delta}, \qquad \eta|_{\Gamma\setminus(\Delta\cup\Lambda_\Delta)} = \xi|_{\Gamma\setminus(\Delta\cup\Lambda_\Delta)}.$$

For any finite $\Lambda' \supset \Lambda_\Delta$, define $\tilde{\omega} := r_{\Lambda',\,\Lambda_\Delta}(\eta)$ and

$$a_\Delta(\omega_\Delta) := f_{\Lambda'}(\tilde{\omega}).$$

We must show that this function is well defined.

Fix $\Lambda' \supset \Lambda_\Delta$. If two extensions $\tilde{\eta}, \tilde{\eta}' \in \Omega_{\Gamma\setminus\Lambda_\Delta}$ both have the same projection $r_{\Lambda',\Lambda_\Delta}(\tilde{\eta}) = r_{\Lambda',\Lambda_\Delta}(\tilde{\eta}') \equiv \tilde{\omega} \in \Omega_{\Gamma\setminus\Lambda'}$, then $f_{\Lambda_\Delta}(\tilde{\eta}) = f_{\Lambda_\Delta}(\tilde{\eta}') = f_{\Lambda'}(\tilde{\omega})$ by the consistency relation (20). Second, if $\Lambda, \Lambda' \supset \Lambda_\Delta$, and $\eta \in \Omega_{\Gamma\setminus\Lambda_\Delta}$ is an extension of $\Omega_{\Gamma\setminus(\Lambda\cup\Lambda')}$, write $\tilde{\omega} := r_{\Lambda\cup\Lambda',\Lambda_\Delta}(\tilde{\eta})$, then again by consistency relation (20)

$$f_{\Lambda'}(\tilde{\omega}) = f_{\Lambda\cup\Lambda'}(r_{\Lambda\cup\Lambda',\Lambda_\Delta}(\tilde{\eta})), \qquad f_{\Lambda}(\tilde{\omega}) = f_{\Lambda\cup\Lambda'}(r_{\Lambda\cup\Lambda',\Lambda_\Delta}(\tilde{\eta})).$$

Hence the function does not depend on the choice of $\Lambda' \supset \Lambda_\Delta$.

Each $f_{\Lambda'}$ satisfies $\|f_{\Lambda'}\|_\infty \le \|f\|_\infty$, so $\|a_\Delta\|_\infty \le \|f\|_\infty$ for all Δ. Thus the family $(a_\Delta)_{\Delta\Subset\Gamma}$ is uniformly bounded.

For each $\Delta \Subset \Gamma$, a_Δ depends only on coordinates in

$$K_\Delta := \Delta \setminus \Lambda_\Delta.$$

Let $b \in \mathcal{A}_{\Lambda'}$ be any local observable supported in a finite $\Lambda' \Subset \Gamma$. We may take Λ_Δ in such a way that $\Lambda_\Delta \supset \Lambda'$, whenever $\Delta \supset \Lambda'$. Then one has $K_\Delta \cap \Lambda' = \varnothing$, so a_Δ and b have disjoint supports.

To prove uniqueness of the construction, assume the same reference configuration $\xi \in \Omega$ was used in both constructions, and let

$$a = (a_\Delta)_{\Delta\Subset\Gamma}, \qquad a' = (a'_\Delta)_{\Delta\Subset\Gamma}$$

be two families obtained from the same $\mathscr{T}_\infty$–measurable function f by the above procedure (using the same ξ). For each finite $\Lambda' \Subset \Gamma$ denote by $f_{\Lambda'}$

and $g_{\Lambda'}$ the corresponding $\mathscr{F}_{\Gamma\setminus\Lambda'}$–measurable representatives of f obtained in the two constructions. Hence, for every $y \in \Omega_{\Gamma\setminus\Lambda'}$ pick any $\omega \in \Omega$ with $\pi_{\Gamma\setminus\Lambda'}(\omega) = y$ and obtain

$$f_{\Lambda'}(y) = f(\omega) = g_{\Lambda'}(y),$$

i.e. $f_{\Lambda'} = g_{\Lambda'}$ as functions on $\Omega_{\Gamma\setminus\Lambda'}$. Now fix $\Delta \Subset \Gamma$ and $\omega_\Delta \in \Omega_\Delta$. Form the partial configuration $\eta \in \Omega_{\Gamma\setminus\Lambda_\Delta}$ as in the construction, for any finite $\Lambda' \supset \Lambda_\Delta$ choose $\tilde{\omega} = r_{\Lambda',\,\Lambda_\Delta}(\eta)$; then by the definition of a_Δ, a'_Δ we have

$$a_\Delta(\omega_\Delta) = f_{\Lambda'}(\tilde{\omega}), \qquad a'_\Delta(\omega_\Delta) = g_{\Lambda'}(\tilde{\omega}).$$

Since $f_{\Lambda'} = g_{\Lambda'}$ on $\Omega_{\Gamma\setminus\Lambda'}$ it follows that

$$a_\Delta(\omega_\Delta) = a'_\Delta(\omega_\Delta).$$

As ω_Δ was arbitrary, $a_\Delta = a'_\Delta$ for every $\Delta \Subset \Gamma$, and hence the two families coincide.

$\square$

Further Research

This work opens new avenues for future research in this field. Some potential directions are summarized below.

- Investigate the state space of the non-commutative C^*-algebra $[\mathcal{C}]^\infty$, in particular whether it forms a Choquet simplex (Bauer, Poulsen, etc.) and characterize extremal states via a suitable notion of ergodicity [5].
- Study the dynamics of the algebra, including automorphisms and strong continuity, with applications to quantum spin systems where each local C^*-algebra is a matrix algebra.
- Explore the Kubo-Martin-Schwinger (KMS) condition for $[\mathcal{B}]^\infty_\gamma$, the existence and properties of KMS states, and their relation to translation-invariant states on the quasi-local algebra [3, 4, 11].
- For the commutative C^*-algebra $[\mathcal{D}]^\infty$, examine the classical KMS condition via the Poisson bracket and compare it with the Dobrushin-Lanford-Ruelle (DLR) equilibrium condition [8, 6].

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